

The nonlinear steepest descent approach to the asymptotics of the second Painlevé transcendent in the complex domain

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Abstract

The asymptotics of the generic second Painlevé transcendent $u(x)$ as $|x| \rightarrow \infty$, $\arg x \in (\frac{\pi}{3}k, \frac{\pi}{3}(k+1))$, $k = 0, 1, \dots, 5$ is found and justified via the direct asymptotic analysis of the associated Riemann-Hilbert problem based on the Deift-Zhou nonlinear steepest descent method. The asymptotics is proved of the Boutroux type, i.e. it is expressed in terms of the elliptic functions. Kapaev-Novokshenov's explicit connection formulae between the asymptotic phases in the different sectors are obtained as well.

1 Introduction

The large argument asymptotic behavior of the Painlevé functions of the first and second kinds were first described in the classical work of P. Boutroux [1]. In particular, Boutroux found a general 2-parameter elliptic (i.e. expressed in terms of elliptic functions) asymptotic formulae for the Painlevé functions.

In 1988, using the multiscale expansion method, N. Joshi and M. Kruskal [16] revised the result of Boutroux and found the phase shift in the Boutroux elliptic ansatz. Specifically, Joshi and Kruskal showed that in the case of the second Painlevé equation,

$$\frac{d^2 u}{dx^2} = 2u^3 + xu, \quad (1.1)$$

the large $|x|$ asymptotics of its generic solution $u(x)$ is given by the following equation,

$$u(x) \sim c_1 \sqrt{x} \operatorname{sn} \left(\frac{2}{3} c_2 x^{3/2} + \mu \omega_1 + \nu \omega_2 |D| \right). \quad (1.2)$$

Here the periods ω_j , $j = 1, 2$ of the elliptic sine, the multipliers $c_{1,2}$, and the module D are (transcendental) functions of $\arg x$, while the parameters μ, ν are constant in the interior of any sector $\arg x \in (\frac{\pi}{3}k, \frac{\pi}{3}(k+1))$, $k = 0, 1, \dots, 5$.

The connection formulae between the constants μ, ν in the ansatz (1.2) corresponding to the different sectors were found by V. Novokshenov [26] and A. Kapaev [20, 21] with the help of the isomonodromy deformation method

which was introduced in the very beginning of the 80-s by H. Flaschka and A. Newell [9] and by M. Jimbo, T. Miwa and K. Ueno [17, 18, 19].

The isomonodromy deformation method associates with a given Painlevé equation an auxiliary system of linear ODEs with rational coefficients whose monodromy data are the first integrals of the Painlevé equation. Hence the asymptotic evaluation of the solutions of the Painlevé equations reduces to the asymptotic solution either of the direct or of the inverse monodromy problems for the auxiliary linear system. The relevant asymptotic scheme based on the asymptotic analysis of the direct monodromy problem was suggested in [14] and was used there and in the subsequent works of several other authors for the evaluation of the connection formulae for the Painlevé transcendent. We refer the reader to review [12] for more on the history and apparatus of the isomonodromy method.

The approach of [26] and [20, 21] is based on an extension of the direct monodromy problem technique of [14] to the asymptotic analysis of the Painlevé functions in the complex domain. The further development of this scheme was made in the works of A. Kitaev [22] and A. Kapaev and A. Kitaev [23] which were also devoted to the elliptic asymptotics of the Painlevé transcendents.

In the beginning of the 90s, P. Deift and X. Zhou proposed a nonlinear version of the classical steepest descent method for oscillatory Riemann-Hilbert problems [5]. This allowed to develop an alternative to [14] asymptotic approach for solving the Painlevé equations based on the asymptotic solution of the relevant *inverse* monodromy problems, which can be formulated as the oscillatory matrix Riemann-Hilbert (RH) problems. This approach has the advantage of not using any prior information of the solutions of the Painlevé equations, and it has already been applied to the trigonometric and exponentially decreasing asymptotics of the Painlevé functions on the real axes [6, 15, 13].

The goal of this paper is to extend the Deift-Zhou method, in the case of the second Painlevé equation (1.1), to the complex domain. As a result we will obtain, rigorously and without any prior assumption crucial for [26] and [20, 21], the Boutroux type elliptic asymptotics together with the explicit formulae for the relevant phase shifts directly from the associated Riemann-Hilbert problem.

The plan of the paper is as follows. In Section 2, we remind the setting the RH problem related to equation (1.1) and formulate our main theorem (Theorem 2.1). In Section 3, we transform the RH problem to the one posed on the graph consisting of the anti-Stokes lines of a certain function $g(z)$ emanating from its critical points and connected by three finite Stokes lines. In Section 4, we explicitly construct the function $g(z)$ mentioned¹. In Section 5, we construct the solution of the reduced RH problem on the finite Stokes lines of $g(z)$ in terms of the Jacobi theta functions. In Section 6, we construct the approximate solutions of the RH problem near the critical points of $g(z)$ in terms of the Airy functions. In Section 7, we collect all the functions together and prove that

¹In the most recent applications of the nonlinear steepest descent method to the orthogonal polynomials (see e.g. [4]) this step corresponds to the construction of the so called equilibrium measure

this combined function approximates the solution of the original RH problem. In Section 8, we extract from the approximate solution of the RH problem the approximate solution of the second Painlevé equation.

The authors dedicate this paper to Barry McCoy on the occasion of his 60th birthday. We are especially happy to do this since it was a remarkable 1977 paper of McCoy, Tracy, and Wu [25] where it was demonstrated for the very first time that an explicit and complete connection formulae for a Painlevé transcendent are possible.

2 RH parametrization of the Painlevé functions of the second kind. Formulation of the main theorem

According to Flaschka and Newell [9] (see also [8] and [14]) the Riemann-Hilbert problem associated with the second Painlevé equation (1.1) consists of finding the piecewise holomorphic 2×2 matrix-valued function $\Psi(\lambda)$ of the complex variable λ such that

- 1) $\Psi(\lambda)$ is holomorphic for $\lambda \in \mathbb{C} \setminus \cup\{\gamma_k\}$, where γ_k are the rays $\gamma_k = \{\lambda \in \mathbb{C} : \arg \lambda = \frac{\pi}{6} + \frac{\pi}{3}(k-1)\}$, $k = 1, \dots, 6$, oriented from zero to infinity, and

$$\Psi(\lambda)e^{\theta(\lambda,x)\sigma_3} \rightarrow I, \quad \lambda \rightarrow \infty, \quad \theta(\lambda,x) = i\left(\frac{4}{3}\lambda^3 + x\lambda\right), \quad (2.1)$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

- 2) on the rays γ_k the jump conditions hold

$$\Psi_+(\lambda) = \Psi_-(\lambda)S_k, \quad \lambda \in \gamma_k, \quad (2.2)$$

where

$$S_{2k-1} = \begin{pmatrix} 1 & 0 \\ s_{2k-1} & 1 \end{pmatrix}, \quad S_{2k} = \begin{pmatrix} 1 & s_{2k} \\ 0 & 1 \end{pmatrix}, \quad (2.3)$$

and the parameters s_k do not depend neither on x nor on λ and satisfy the constraints

$$s_{k+3} = -s_k, \quad s_1 - s_2 + s_3 + s_1s_2s_3 = 0. \quad (2.4)$$

Here, $\Psi_+(\lambda)$ and $\Psi_-(\lambda)$ are the respective limits of $\Psi(\lambda)$ to the left and to the right of γ .

The RH problem is depicted in Figure 1.

It is worth noticing that the first of the relations (2.4) and the central symmetry of the RH problem graph imply the symmetry equation

$$\Psi(\lambda) = \sigma_2 \Psi(-\lambda) \sigma_2, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad (2.5)$$

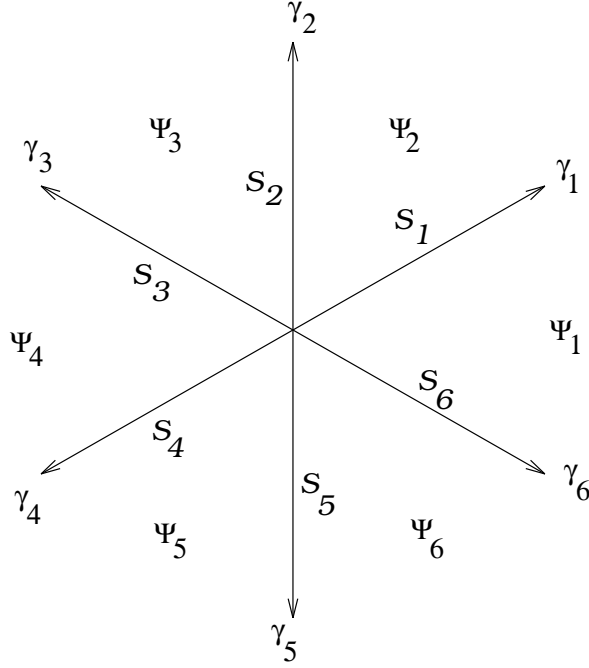


Figure 1: The RH problem graph γ for the Ψ -function.

Also, the special triangular structure of the matrices S_k ensures that the product $e^{-\theta\sigma_3}S_k e^{\theta\sigma_3} \rightarrow I$ as $\lambda \rightarrow \infty$, i.e. the well-posedness of the RH problem.

The piecewise holomorphic function $\Psi(\lambda)$ is in fact a collection of the canonical fundamental solutions of the linear 2×2 matrix ODE

$$\frac{d\Psi}{d\lambda} = \{-i(4\lambda^2 + x + 2u^2)\sigma_3 - 4u\lambda\sigma_2 - 2v\sigma_1\}\Psi, \quad (2.6)$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where v and u are (meromorphic) functions of x such that $v = u_x$ and $u(x)$ satisfies the second Painlevé equation (1.1). The jump matrices S_k of the RH problem have the meaning of the Stokes matrices for (2.6) and, by construction, do not depend on x . Therefore, the second Painlevé equation (1.1) describes the isomonodromy deformations of system (2.6).

The Painlevé function $u(x)$ admits the following representation in terms of the solution $\Psi(\lambda)$ of the RH problem,

$$u(x) = 2 \lim_{\lambda \rightarrow \infty} \left(\lambda \Psi_{12}(\lambda) e^{-\theta(\lambda, x)} \right). \quad (2.7)$$

Moreover, the respective map,

$$\{s\} \equiv \{s_k : s_{k+3} = -s_k, \quad s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0\} \mapsto \{\text{solutions of (1.1)}\},$$

is a bijection. Hence we adopt the notation

$$u(x) \equiv u(x|s)$$

for the second Painlevé transcendent.

The following important symmetry properties take place,

$$u(x|s) = e^{i\frac{2\pi}{3}l} u(xe^{i\frac{2\pi}{3}l} | s^{2l}), \quad (2.8)$$

$$u(x|s) = \bar{u}(\bar{x}|s^*), \quad (2.9)$$

where $s \mapsto s^d$, $d \in \mathbb{Z}$ and $s \mapsto s^*$ denote the automorphisms of the set of monodromy data $\{s\}$ defined by the relations

$$(s^d)_j := s_{j+d},$$

and

$$(s^*)_j := \bar{s}_{4-j}.$$

Symmetries (2.8) and (2.9) show that to obtain a complete asymptotic description of the function $u(x|s)$ as $x \rightarrow \infty$, $x \in \mathbb{C}$, it is enough to find, in terms of s , the asymptotic behavior just in one of the sectors $(\frac{\pi}{3}k, \frac{\pi}{3}(k+1))$, say in the sector $(\frac{2\pi}{3}, \pi)$. The asymptotics of $u(x|s)$ in this sector is then of our principal interest.

Let us introduce a complex valued function $\varkappa = \varkappa(\varphi)$, $0 < \varphi < \frac{\pi}{3}$, which satisfies the system of transcendent equations

$$\begin{aligned} \operatorname{Re} \left[\left(\frac{e^{i\varphi}}{1 + \varkappa^2} \right)^{3/2} (-2\varkappa^2 K' + (1 + \varkappa^2)E') \right] &= 0, \\ \operatorname{Im} \left[\left(\frac{e^{i\varphi}}{1 + \varkappa^2} \right)^{3/2} ((\varkappa^2 - 1)K + (1 + \varkappa^2)E) \right] &= 0, \end{aligned} \quad (2.10)$$

where K and E are the standard complete elliptic integrals

$$K = K(\varkappa) = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-\varkappa^2 z^2)}}, \quad K' = K(\varkappa'),$$

$$E = E(\varkappa) = \int_0^1 \sqrt{\frac{1-\varkappa^2 z^2}{1-z^2}} dz, \quad E' = E(\varkappa'),$$

$$\varkappa' = \sqrt{1-\varkappa^2}, \quad 0 < \arg \varkappa' < \frac{\pi}{2}, \quad -\frac{\pi}{2} < \arg(1+\varkappa^2) < 0.$$

One can show that system (2.10) has a unique smooth solution normalized by the conditions,

$$\phi \rightarrow \frac{2\pi}{3} \Rightarrow \varkappa \rightarrow 1, \quad \phi \rightarrow \pi \Rightarrow \varkappa \rightarrow 0.$$

$$-\frac{\pi}{2} < \arg \varkappa < 0.$$

The main result concerning the asymptotics of $u(x|s)$ for complex x can be now formulated as the following theorem.

Theorem 2.1 *Let*

$$s_3 \neq 0, \quad 1 - s_1 s_3 \neq 0. \quad (2.11)$$

Then

$$u(x|s) = -\frac{i\varkappa}{\sqrt{1+\varkappa^2}} x^{1/2} \operatorname{sn} \left(\frac{2i}{3} \frac{x^{3/2}}{\sqrt{1+\varkappa^2}} - \frac{2iK}{\pi} \ln(s_3) - \frac{K'}{\pi} \ln(1 - s_1 s_3) + \mathcal{O}(x^{-3/2}) \middle| \varkappa \right), \quad (2.12)$$

$$x \rightarrow \infty, \quad \arg x = \varphi + \mathcal{O}(x^{-3/2}), \quad \varphi \in \left(\frac{2\pi}{3}, \pi \right), \quad (2.13)$$

and the asymptotic is uniform on the cheese-type domain

$$\{x \in \mathbb{C}: \quad |\arg x - \varphi| < C|x|^{-3/2}\} \cap D_\varepsilon,$$

where D_ε is the complement to the union of the ε -neighborhoods of all the poles of the indicated elliptic function.

This theorem is due to V. Novokshenov [26] and A. Kapaev [20]. Formulae (2.10) and (2.12) (without the indicated expression for the phase) constitute the classical Boutroux ansatz. In this paper we present a new proof of this theorem based on the Deift-Zhou nonlinear steepest descent method. This proof is free from any prior assumptions, which were crucial for [26] and [20], including the assumption of the solvability of the RH problem under the conditions (2.11). The latter we will obtain *en route*.

We complete this section by noticing that Theorem 2.1 together with the symmetry relations (2.8) and (2.9) yield the following asymptotic description for the rest of the sectors.

Corollary 2.1 *If $s_{2+2l} \neq 0$ and $1 + s_{2+2l}s_{3+2l} \neq 0$ for $l = 0, \pm 1$, then the leading asymptotic term of the Painlevé function in the interior of the sector $\arg x \in (-\frac{2\pi}{3}l; \frac{\pi}{3} - \frac{2\pi}{3}l)$ is elliptic.*

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The phase shift of the elliptic asymptotics in each of the sectors is given by the expression,

$$\frac{2iK}{\pi} \ln s_K \pm \frac{K'}{\pi} \ln s_{K'},$$

where the choice of the monodromy parameters s_K and $s_{K'}$ is indicated in Figure 2. Together with the monodromy equation (2.4) this constitutes the connection formulae for the Painlevé function $u(x|s)$ in the complex domain.

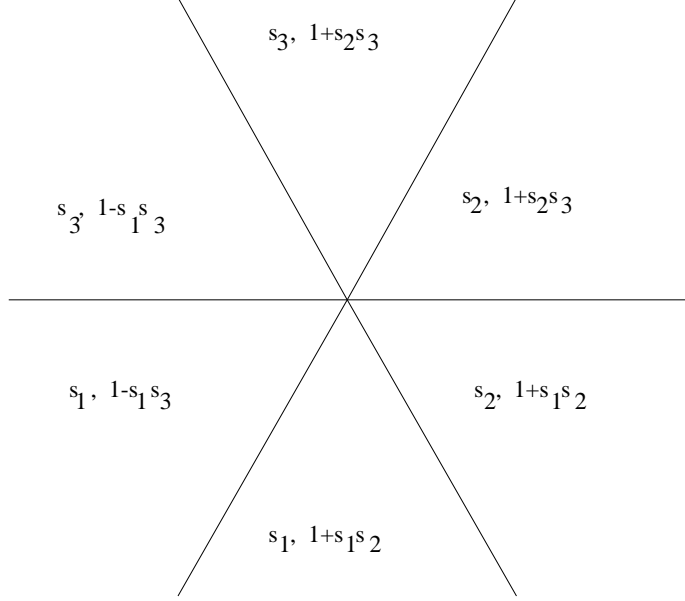


Figure 2: Combinations of the Stokes multipliers whose non-triviality yields the elliptic asymptotic behavior of Painlevé function in the corresponding sector

Remark 2.1 *A complete description of the asymptotic behavior of $u(x|s)$ which includes the “Stokes rays” $\arg x = \frac{\pi}{3}k$ is given in [21]. In fact, the complete description of the asymptotic behavior of the solutions of the second Painlevé equation obtained in [21] covers the general case of the latter, i.e. the equation*

$$\frac{d^2u}{dx^2} = 2u^3 + xu + \alpha, \quad \alpha \in \mathbb{C}. \quad (2.14)$$

3 The transformation of the RH problem

Let us scale the independent variable λ according to the equation

$$\lambda = (e^{-i\varphi}x)^{1/2}z, \quad (e^{-i\varphi}x)^{3/2} = t, \quad \varphi = \text{const} \in \left(\frac{2\pi}{3}, \pi\right), \quad (3.1)$$

$$\text{Re } t \rightarrow +\infty, \quad |\text{Im } t| < \text{const}.$$

The stationary points of the phase $\theta = it\left(\frac{4}{3}z^3 + e^{i\varphi}z\right)$ are independent of t , and formally, the problem can be treated in the very same way as the RH problem which is dealt with in ref. [6, 15] where the case $x \rightarrow -\infty$, i.e. $\varphi = \pi$, was considered. Indeed, applying literally the same arguments as in ref. [15], one can easily replace rays γ_k by the anti-Stokes lines $\gamma_k^\pm = \{\text{Im } \theta = 0\}$ indicated in Figure 3. Points z_\pm are the stationary phase points, $z_\pm = \mp \frac{i}{2}e^{i\varphi/2}$, and, unlike

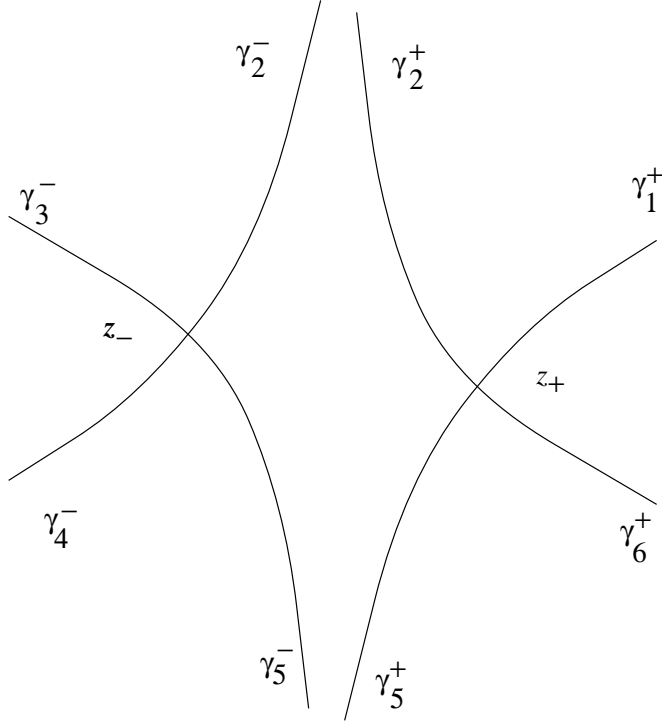


Figure 3: The anti-Stokes lines for the phase $\theta(z) = it(\frac{4}{3}z^3 + e^{i\varphi}z)$.

the case $\varphi = \pi$, the exponent $\theta(z)$ has nonzero real parts at $z = z_{\pm}$:

$$\operatorname{Re} \theta(z_{\pm}) \rightarrow \pm \frac{t}{3} \cos \frac{3\varphi}{2}, \quad t \rightarrow +\infty;$$

since $\varphi \in (\frac{2}{3}\pi, \pi)$, for sufficiently large t ,

$$\operatorname{Re} \theta(z_+) < 0, \quad \operatorname{Re} \theta(z_-) > 0. \quad (3.2)$$

Unlike the case $\varphi = \pi$, now, after the normalization of the RH problem (i.e. $\Psi \rightarrow Y = \Psi e^{\theta(z)\sigma_3}$), only *half* of the jump matrices, namely the matrices corresponding to the lines $\gamma_1^+, \gamma_2^-, \gamma_4^-, \gamma_5^+$, approach the identity as $t \rightarrow \infty$ *uniformly* for $z \in \gamma_k^{\pm}$. The other half of the jump matrices, i.e. the matrices associated with the lines $\gamma_2^+, \gamma_3^-, \gamma_5^-, \gamma_6^+$, “explode” as $t \rightarrow \infty$ in the finite neighborhoods of the points z_{\pm} . Therefore, we have no reason to expect that the RH problem under consideration can be reduced to the Weber-Hermite model RH problems associated to the stationary phase points z_{\pm} . Thus, the principal idea of the asymptotic analysis in the case $\varphi = \pi$ fails as $\varphi \in (\frac{2\pi}{3}, \pi)$.

A regular procedure that allows us to overcome the indicated above obstacles is based on the following construction. Firstly, we “split” the stationary phase

points z_{\pm} into four new points

$$z_1, z_2 = -z_1, z_3, z_4 = -z_3,$$

where z_1, z_3 lie in a neighborhood of the stationary phase point z_+ , while z_2, z_4 are associated in the same way with the stationary point z_- . Secondly, we assume that there exists an analytical function $g(z)$, which satisfies the conditions:

- a) $g(z) = i(\frac{4}{3}z^3 + e^{i\varphi}z) + \mathcal{O}(\frac{1}{z})$, as $z \rightarrow \infty$;
- b) points $\pm z_1, \pm z_3$ are the branch points of $g'(z)$ of order 2 and, in their neighborhoods,

$$g'(z) = c_{1,3}^{\mp}(z \pm z_{1,3})^{1/2} + \dots,$$

with $c_{1,3}^{\mp} \neq 0$.

- c) $\operatorname{Re} g(\pm z_{1,3}) = 0$.
- d) $g' \neq 0$ on $\mathbb{C} \setminus \{\pm z_{1,3}\}$.

In fact, conditions a)–d) constitute a certain problem of the theory of analytic functions. Simultaneously, they are the restrictions on the points $\pm z_{1,3}$ themselves. In Section 4, we will solve the problem by the use of an elliptic integral associated with the elliptic curve $w^2 = (z^2 - z_1^2)(z^2 - z_3^2)$. We see then that condition c) is just the classical Boutroux equations, which always appear in the asymptotic analysis of the Painlevé equations. It also should be emphasizing that a flexibility we gain introducing four parameters, i.e. points $\pm z_{1,3}$, plays the crucial role for solvability of the problem a)–d).

Having assumed the existence of the function $g(z)$, our main idea is to replace the original phase $\theta = it(\frac{4}{3}z^3 + e^{i\varphi}z)$ by the function $g(z)$. Because of the asymptotic equation a), this replacement preserves the normalization condition (2.1). Moreover, due to condition c), we eliminate the main trouble, i.e. inequalities (3.2). Nevertheless, there is a price: now, we have to deform the initial RH problem to the RH problem formulated for the anti-Stokes lines $\hat{\gamma}^{(k)}$ corresponding to the function $g(z)$:

$$\operatorname{Im} g(z) = \operatorname{Im} g(z_k). \quad (3.3)$$

The anti-Stokes graph defined in (3.3) is more complicated than the θ -graph depicted in Figure 3. Properties a), b) and d) lead to the g -anti-Stokes lines $\hat{\gamma}_s^{(k)}$ as shown in Figure 4. Our main task for this section, is the transformation of the initial RH graph presented in Figure 1 into the one given in Figure 4.

The notation $\hat{\gamma}_s^{(k)}$ for the anti-Stokes line used here refers to its starting point z_k and to the anti-Stokes ray γ_s as being asymptotic for the line considered.

As it is easy to see, the transformation problem is not trivial and can not be reduced to the simple bending of the graph branches due to the very different topological properties of the graphs presented in Figure 1 and Figure 4. In particular, the RH problem graph γ for the Ψ function is connected, while the anti-Stokes graph $\hat{\gamma}$ is disconnected. The most we can do with the original problem by means of trivial bending of the branches is shown in Figure 5.

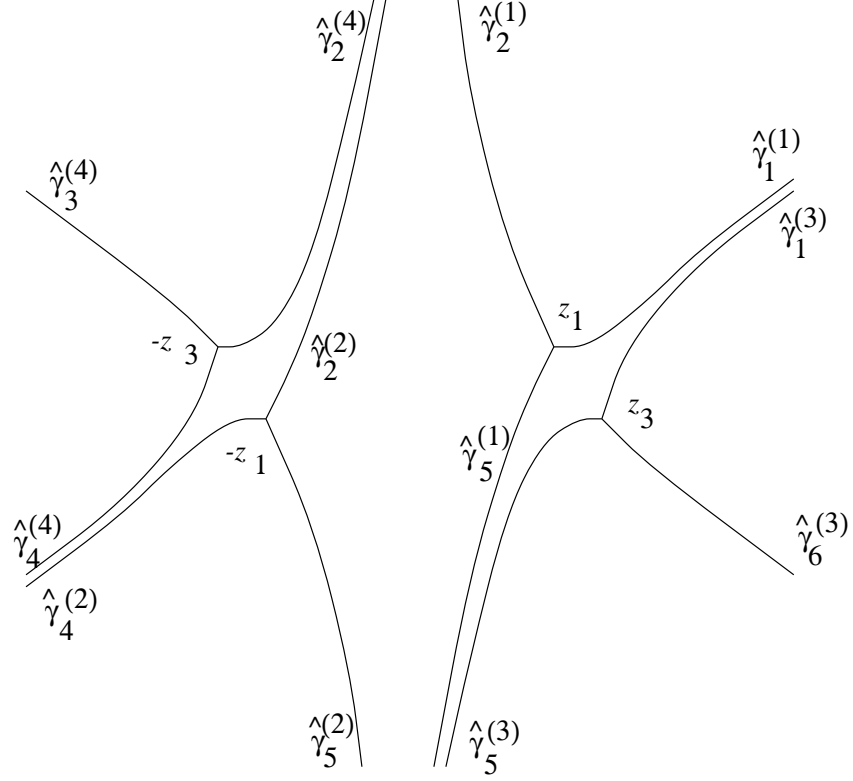


Figure 4: The anti-Stokes lines $\hat{\gamma}$ for the equation for $x \rightarrow \infty$, $\arg x \in (\frac{2\pi}{3}, \pi)$.

Note that after such deformation, a new boundary curve between the third and the sixth domains arises where the jump is described by the ratio of the sixth and the third canonical solutions:

$$\Psi_6(z) = \Psi_3(z) S_3 S_4 S_5, \quad (3.4)$$

where

$$S_3 S_4 S_5 = \begin{pmatrix} 1 + s_1 s_2 & -s_1 \\ -s_1 & 1 - s_1 s_3 \end{pmatrix},$$

Let us consider the function $\delta(z)$ depending on z_k , $k = 1, 3$, and some parameters ν_l , $l = 1, 2, 3$,

$$\delta(z) = \left(\frac{z + z_1}{z + z_3} \right)^{\nu_3} \left(\frac{z - z_1}{z + z_1} \right)^{\nu_2} \left(\frac{z - z_3}{z - z_1} \right)^{\nu_1}, \quad (3.5)$$

which is defined on the complex plane cut along the broken line $[z_3, z_1] \cup [z_1, -z_1] \cup [-z_1, -z_3]$. The function is fixed by condition

$$\delta(z) \xrightarrow{z \rightarrow \infty} 1, \quad (3.6)$$

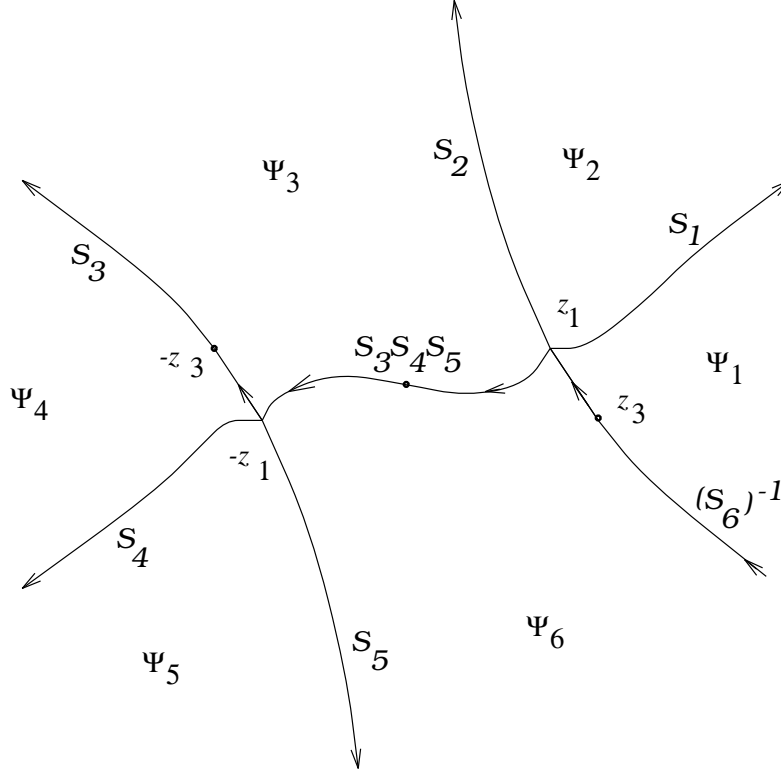


Figure 5: The continuously deformed RH problem graph.

and has the jumps on the cuts

$$\begin{aligned}
 \delta_-(z) &= \delta_+(z)e^{2\pi i\nu_1}, & z \in (z_3, z_1), \\
 \delta_-(z) &= \delta_+(z)e^{2\pi i\nu_2}, & z \in (z_1, -z_1), \\
 \delta_-(z) &= \delta_+(z)e^{2\pi i\nu_3}, & z \in (-z_1, -z_3),
 \end{aligned} \tag{3.7}$$

as it is shown in Figure 6.

The next step is the most important for the transformation of the original RH problem. Let us introduce the new function

$$\hat{\Psi} = \Psi \delta^{-\sigma_3}. \tag{3.8}$$

This new function does not differ from the original one at infinity, but its jumps are described by the new matrices:

$$\hat{\Psi}_+ = \hat{\Psi}_- \delta_-^{\sigma_3} S \delta_+^{-\sigma_3} = \hat{\Psi}_- \hat{S}. \tag{3.9}$$

The new RH graph is shown in Figure 7, and the new jump matrices \hat{S} are given by the equations,

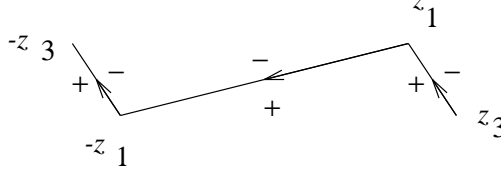


Figure 6: The graph of the scalar RH problem solved by the function $\delta(z)$.

$$\begin{aligned}\hat{S}_{2k-1} &= \begin{pmatrix} 1 & 0 \\ s_{2k-1}\delta^{-2} & 1 \end{pmatrix}, \quad \hat{S}_{2k} = \begin{pmatrix} 1 & s_{2k}\delta^2 \\ 0 & 1 \end{pmatrix}, \\ \check{S}_3 &= \begin{pmatrix} e^{2\pi i\nu_3} & 0 \\ s_3\delta_-^{-2}e^{2\pi i\nu_3} & e^{-2\pi i\nu_3} \end{pmatrix}, \quad (\check{S}_6)^{-1} = \begin{pmatrix} e^{2\pi i\nu_1} & s_3\delta_-^2e^{-2\pi i\nu_1} \\ 0 & e^{-2\pi i\nu_1} \end{pmatrix}, \\ \widehat{S_3 S_4 S_5} &= \begin{pmatrix} (1 + s_1 s_2)e^{2\pi i\nu_2} & -s_1\delta_-^2e^{-2\pi i\nu_2} \\ -s_1\delta_-^{-2}e^{2\pi i\nu_2} & (1 - s_1 s_3)e^{-2\pi i\nu_2} \end{pmatrix}.\end{aligned}\tag{3.10}$$

Now, we are going to explain what this change is made for. The point is that some of the new jump matrices can be factorized in such a way, that the simple transformation of the RH graph allows us to approach the anti-Stokes lines indicated in Figure 4 (cf. [15]). Indeed, let us assume that the generic conditions hold true,

$$s_3 \neq 0, \quad 1 - s_1 s_3 \neq 0.\tag{3.11}$$

If we then put

$$e^{-2\pi i\nu_3} = s_3,\tag{3.12}$$

then it is easy to check the equation holds:

$$\check{S}_3 = \hat{S}_2^U \hat{S}_A \hat{S}_4^U,\tag{3.13}$$

where

$$\hat{S}_2^U = \begin{pmatrix} 1 & \delta_-^2 s_3^{-1} \\ 0 & 1 \end{pmatrix}, \quad \hat{S}_A = \begin{pmatrix} & -\delta_-^2 \\ \delta_-^{-2} & \end{pmatrix}, \quad \hat{S}_4^U = \begin{pmatrix} 1 & \delta_+^2 s_3^{-1} \\ 0 & 1 \end{pmatrix}.\tag{3.14}$$

Similarly, if we put

$$e^{2\pi i\nu_1} = s_3, \quad \text{or} \quad \nu_1 = -\nu_3,\tag{3.15}$$

then the factorization arises

$$\check{S}_6^{-1} = \hat{S}_1^L \hat{S}_B \hat{S}_5^L,\tag{3.16}$$

with the matrices

$$\hat{S}_1^L = \begin{pmatrix} 1 & 0 \\ (\delta_-^2 s_3)^{-1} & 1 \end{pmatrix}, \quad \hat{S}_B = \begin{pmatrix} & \delta_-^2 \\ -\delta_-^{-2} & \end{pmatrix}, \quad \hat{S}_5^L = \begin{pmatrix} 1 & 0 \\ (\delta_+^2 s_3)^{-1} & 1 \end{pmatrix}.\tag{3.17}$$

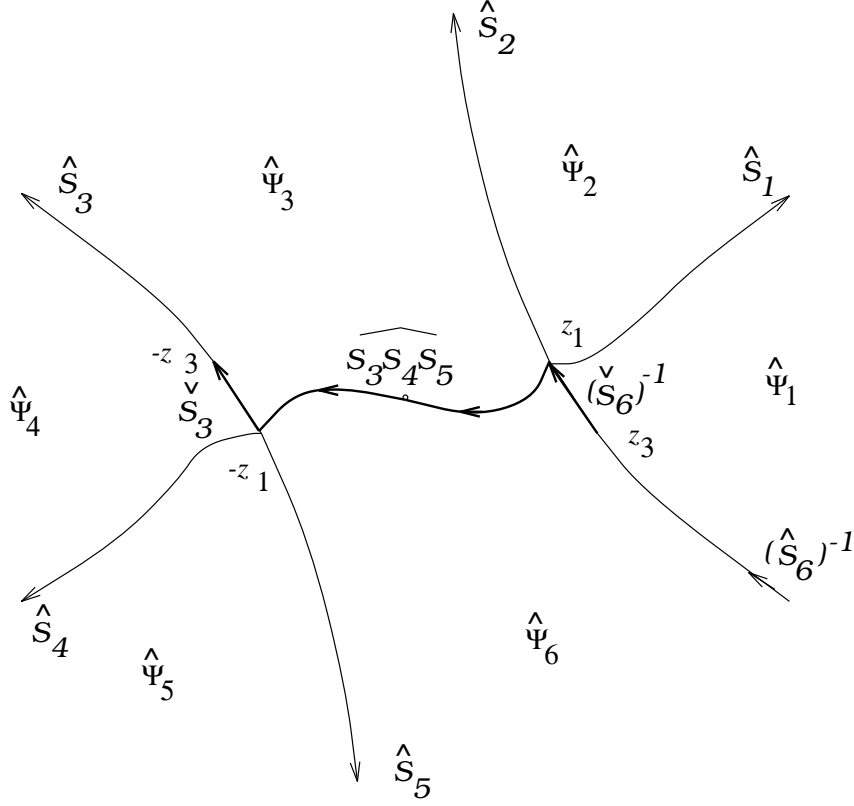


Figure 7: The RH problem for $\hat{\Psi}$.

At the same time, if we put

$$e^{2\pi i\nu_2} = 1 - s_1 s_3, \quad (3.18)$$

then

$$\widehat{S_3 S_4 S_5} = \hat{S}^U \hat{S}^L \quad (3.19)$$

with

$$\hat{S}^U = \begin{pmatrix} 1 & -\frac{s_1}{1-s_1 s_3} \delta_-^2 \\ 0 & 1 \end{pmatrix}, \quad \hat{S}^L = \begin{pmatrix} 1 & 0 \\ -\frac{s_1}{1-s_1 s_3} \delta_+^{-2} & 1 \end{pmatrix}. \quad (3.20)$$

The “intermediate” RH graph is shown in Figure 8, and the final graph is presented in Figure 9.

Here, the jump matrices are

$$\hat{S}_5^{(3)} = \begin{pmatrix} 1 & 0 \\ \frac{1}{\delta^2 s_3} & 1 \end{pmatrix}, \quad \hat{S}_6^{-1} = \begin{pmatrix} 1 & \delta^2 s_3 \\ 0 & 1 \end{pmatrix}, \quad \hat{S}_1^{(3)} = \begin{pmatrix} 1 & 0 \\ \frac{1}{\delta^2 s_3} & 1 \end{pmatrix}, \quad (3.21)$$

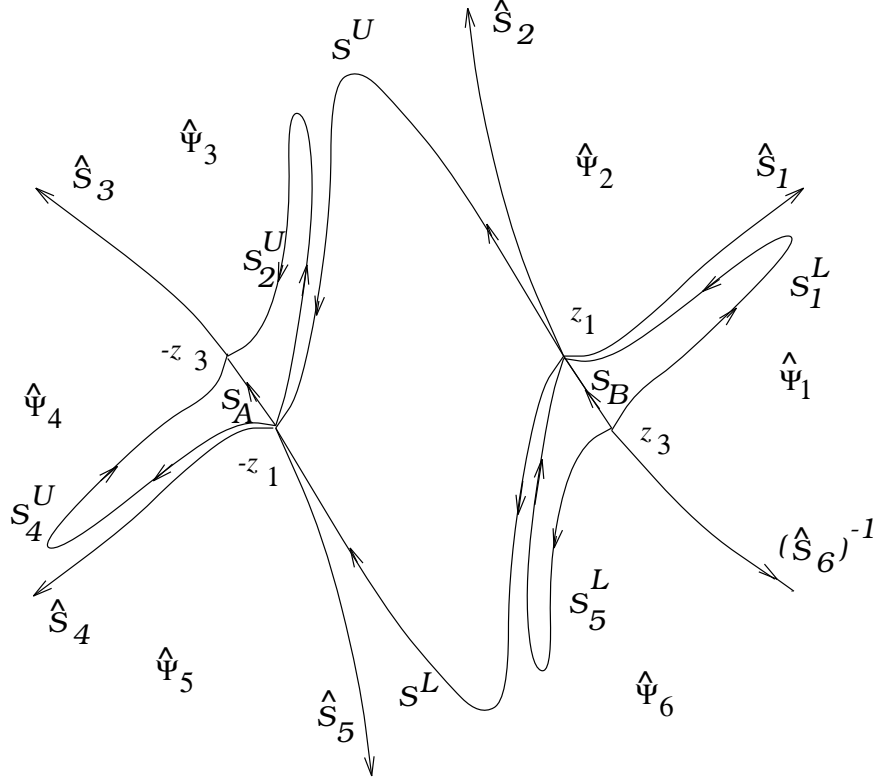


Figure 8: Deformation of the RH problem graph for $\hat{\Psi}$.

$$\hat{S}_1^{(1)} = \begin{pmatrix} 1 & 0 \\ -\frac{1-s_1 s_3}{\delta^2 s_3} & 1 \end{pmatrix}, \quad \hat{S}_2^{(1)} = \begin{pmatrix} 1 & \frac{\delta^2 s_3}{1-s_1 s_3} \\ 0 & 1 \end{pmatrix}, \quad \hat{S}_5^{(1)} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{\delta^2 s_3 (1-s_1 s_3)} & 1 \end{pmatrix}, \quad (3.22)$$

$$\hat{S}_4^{(2)} = \begin{pmatrix} 1 & \delta^2 \frac{1-s_1 s_3}{s_3} \\ 0 & 1 \end{pmatrix}, \quad \hat{S}_5^{(2)} = \begin{pmatrix} 1 & 0 \\ -\frac{s_3}{\delta^2 (1-s_1 s_3)} & 1 \end{pmatrix}, \quad \hat{S}_2^{(2)} = \begin{pmatrix} 1 & \frac{\delta^2}{s_3 (1-s_1 s_3)} \\ 0 & 1 \end{pmatrix}, \quad (3.23)$$

$$\hat{S}_2^{(4)} = \begin{pmatrix} 1 & -\frac{\delta^2}{s_3} \\ 0 & 1 \end{pmatrix}, \quad \hat{S}_3 = \begin{pmatrix} 1 & 0 \\ \frac{s_3}{\delta^2} & 1 \end{pmatrix}, \quad \hat{S}_4^{(4)} = \begin{pmatrix} 1 & -\frac{\delta^2}{s_3} \\ 0 & 1 \end{pmatrix}, \quad (3.24)$$

$$\hat{S}_A = \begin{pmatrix} 1 & -\delta_-^2 \\ \frac{1}{\delta_-^2} & \end{pmatrix}, \quad \hat{S}_B = \begin{pmatrix} -\frac{1}{\delta_-^2} & \delta_-^2 \end{pmatrix}. \quad (3.25)$$

Also, taking into account the equations (3.12), (3.15) and (3.18), we rewrite the jump conditions (3.7) for the function $\delta(z)$ (3.5) as follows,

$$\delta_-(z) = \delta_+(z) s_3, \quad z \in (z_3, z_1),$$

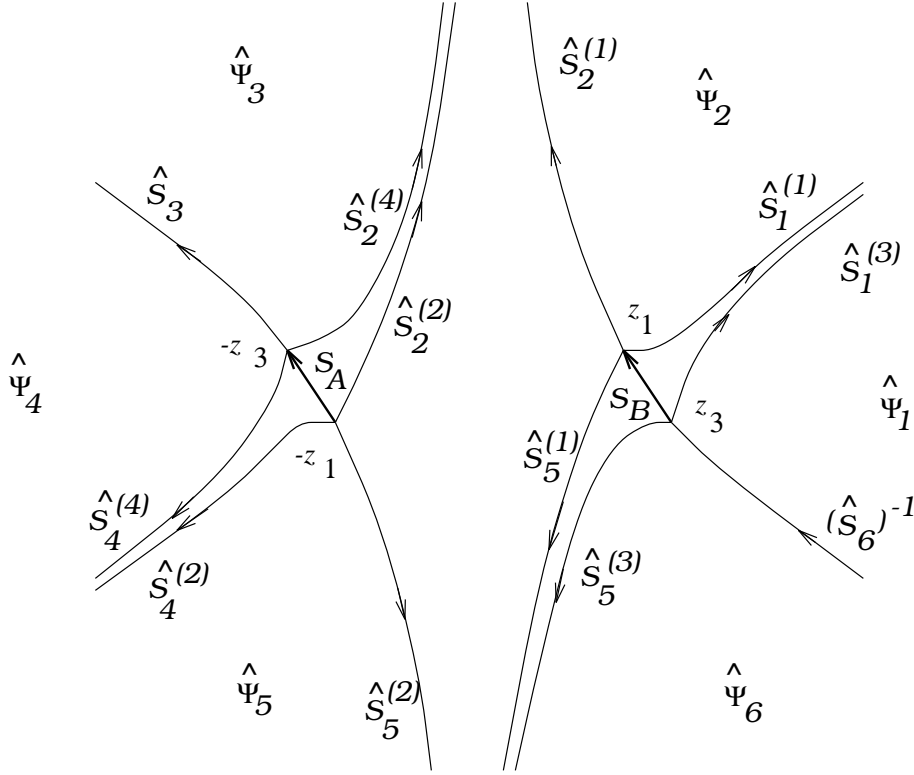


Figure 9: The final RH problem graph for $\hat{\Psi}$.

$$\begin{aligned}\delta_-(z) &= \delta_+(z)(1 - s_1 s_3), & z \in (z_1, -z_1), \\ \delta_-(z) &= \delta_+(z) \cdot \frac{1}{s_3}, & z \in (-z_1, -z_3).\end{aligned}\tag{3.26}$$

In this form, the RH problem graph for the $\hat{\Psi}$ -function coincides with the anti-Stokes graph $\hat{\gamma}$, see Figure 4, except for the additional segments $[z_3, z_1]$ and $[-z_1, -z_3]$.

Now, we do not need the function δ anymore and can return to the original Ψ -function by the use of (3.8),

$$\Psi = \hat{\Psi} \delta^{\sigma_3}.\tag{3.27}$$

The calculations above show that the solution of the second Painlevé equation can be obtained from the Ψ -function with the jumps on the graph shown on Figure 10. The new RH problem, which replaces the problem (2.1)–(2.4), can be formulated as follows.

- 1) Find a piecewise holomorphic matrix-valued 2×2 function Ψ of the complex variable z with an irregular singular point at infinity where it has the

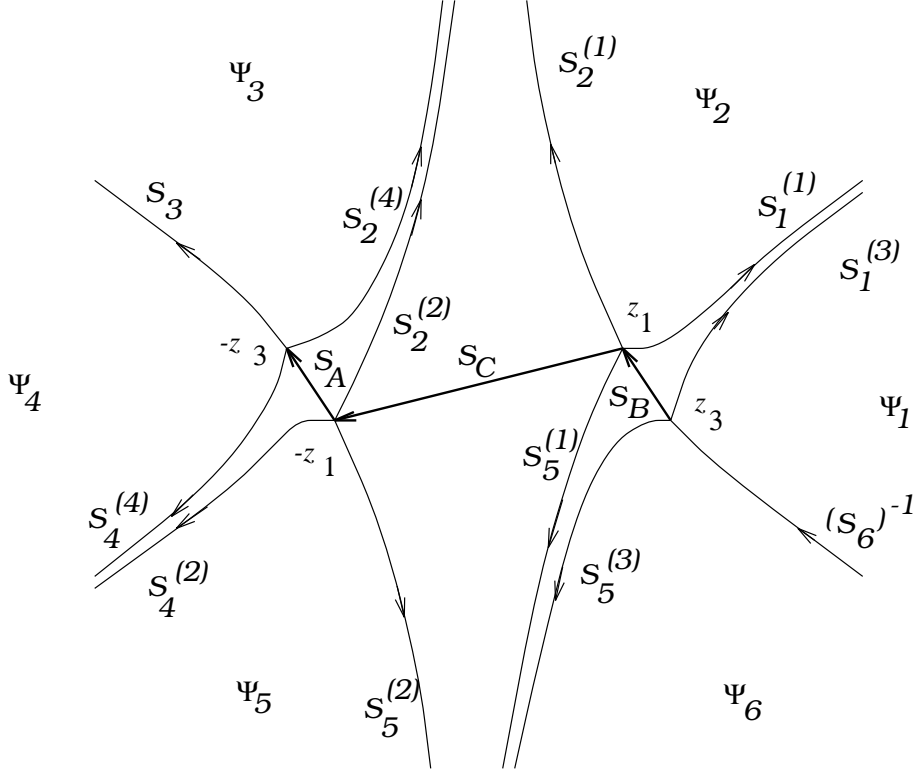


Figure 10: The ultimate RH problem graph for Ψ .

exponential asymptotic behavior of the type (2.1)

$$\Psi(z) = (I + \mathcal{O}(z^{-1}))e^{-t\theta(z)\sigma_3}, \quad \theta(z) = i\left(\frac{4}{3}z^3 + e^{i\varphi}z\right), \quad z \rightarrow \infty; \quad (3.28)$$

2) on the piecewise smooth oriented graph $\hat{\gamma}$, see Figure 10, the jump condition holds true,

$$\Psi_+(z) = \Psi_-(z)S, \quad \lambda \in \hat{\gamma}. \quad (3.29)$$

Here, $\Psi_+(z)$ and $\Psi_-(z)$ are the limits of the function $\Psi(z)$ on the contour $\hat{\gamma}$ to the left and to the right, respectively, and the piecewise constant matrix S is described by the following equations,

$$S_5^{(3)} = \begin{pmatrix} 1 & 0 \\ \frac{1}{s_3} & 1 \end{pmatrix}, \quad S_6^{-1} = \begin{pmatrix} 1 & s_3 \\ 0 & 1 \end{pmatrix}, \quad S_1^{(3)} = \begin{pmatrix} 1 & 0 \\ \frac{1}{s_3} & 1 \end{pmatrix}, \quad (3.30)$$

$$S_1^{(1)} = \begin{pmatrix} 1 & 0 \\ -\frac{1-s_1s_3}{s_3} & 1 \end{pmatrix}, \quad S_2^{(1)} = \begin{pmatrix} 1 & \frac{s_3}{1-s_1s_3} \\ 0 & 1 \end{pmatrix}, \quad S_5^{(1)} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{s_3(1-s_1s_3)} & 1 \end{pmatrix}, \quad (3.31)$$

$$S_4^{(2)} = \begin{pmatrix} 1 & \frac{1-s_1 s_3}{s_3} \\ 0 & 1 \end{pmatrix}, \quad S_5^{(2)} = \begin{pmatrix} 1 & 0 \\ -\frac{s_3}{1-s_1 s_3} & 1 \end{pmatrix}, \quad S_2^{(2)} = \begin{pmatrix} 1 & \frac{1}{s_3(1-s_1 s_3)} \\ 0 & 1 \end{pmatrix}, \quad (3.32)$$

$$S_2^{(4)} = \begin{pmatrix} 1 & -\frac{1}{s_3} \\ 0 & 1 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 \\ s_3 & 1 \end{pmatrix}, \quad S_4^{(4)} = \begin{pmatrix} 1 & -\frac{1}{s_3} \\ 0 & 1 \end{pmatrix}, \quad (3.33)$$

$$S_A = \begin{pmatrix} & -\frac{1}{s_3} \\ s_3 & \end{pmatrix}, \quad S_B = \begin{pmatrix} & s_3 \\ -\frac{1}{s_3} & \end{pmatrix}, \quad S_C = \begin{pmatrix} \frac{1}{1-s_1 s_3} & \\ & 1-s_1 s_3 \end{pmatrix}. \quad (3.34)$$

The parameters s_k satisfy the constraint (2.4),

$$s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0, \quad (3.35)$$

and do not depend neither on t nor on z .

Remark 3.1 *Possibly, given the jump contour $\hat{\gamma}$, the resulting jump conditions (3.30)–(3.34) can be reconstructed directly from the jump conditions (2.3) for the contour γ , see Figure 1, using, instead of the “ghost” function $\delta(z)$, the system of algebraic relations for the matrices $S_k^{(l)}$ and S_k .*

4 Construction of the function $g(z)$

In this section, we construct the function $g(z)$ described in Section 3. Consider the function

$$w(z) = \sqrt{(z^2 - z_1^2)(z^2 - z_3^2)}, \quad z_1^2 + z_3^2 = -\frac{1}{2}e^{i\varphi}, \quad (4.1)$$

defined on the Riemann surface Γ , see Figure 11, pasted of two complex planes cut along the intervals $[z_3, z_1]$ and $[-z_1, -z_3]$, while the branch of the root on the upper sheet is chosen in such a way that

$$w = z^2 - \frac{1}{2}(z_1^2 + z_3^2) + \mathcal{O}(z^{-2}) \quad \text{as } z \rightarrow \infty^+.$$

Here ∞^+ denotes the infinite point of the upper sheet. We choose the basis $\{a, b\}$ of the group $H_1(\Gamma)$ as it is indicated in Figure 11. Introduce the Abelian (elliptic, in fact) integral of the second kind

$$\hat{g}(z) = 4i \int_{0_+}^z w(z) dz. \quad (4.2)$$

We shall consider $\hat{g}(z)$ as a single valued function defined on the complex plane \mathbb{C}^* cut along the sum of the intervals $[z_3, z_1] \cup [z_1, -z_1] \cup [-z_1, -z_3]$, moreover

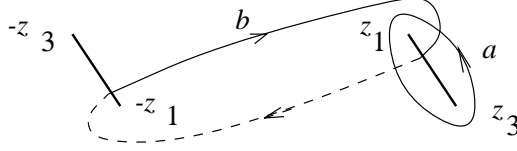


Figure 11: The Riemann surface of the function $w(z)$. a and b mean the basic cycles of the surface.

the starting point $z = 0_+$ of the integration path lies on the left side of the oriented cut $[z_1, -z_1]$. Introduce the periods of (4.2),

$$A = 4i\mathcal{J}_a = 4i \oint_a w(z) dz, \quad B = 4i\mathcal{J}_b = 4i \oint_b w(z) dz. \quad (4.3)$$

Taking into account the symmetry $w(-z) = w(z)$, we find that

$$\hat{g}(z) + \hat{g}(-z) = 4i \left(\int_{0_+}^z + \int_{0_+}^{z_1} + \oint_a + \int_{z_1}^{0_-} + \int_{0_-}^{-z} \right) w(z) dz = A.$$

Thus the function

$$g(z) = \hat{g}(z) - \frac{1}{2}A, \quad (4.4)$$

considered on \mathbb{C}^* , is odd,

$$g(-z) = -g(z).$$

Using definition (4.4), we see that properties b) and d) of $g(z)$ are satisfied.

To find the asymptotics of $g(z)$ as $z \rightarrow \infty$, introduce the regularized integral

$$4i \int_{0_+}^z \left(w(z) - z^2 + \frac{1}{2}(z_1^2 + z_3^2) \right) dz = \hat{g}(z) - i \left(\frac{4}{3}z^3 + e^{i\varphi}z \right).$$

On the other hand, again using the symmetry $w(z) = w(-z)$, it is equal to

$$\begin{aligned} & 2i \left(\int_{-z}^{0_-} + \int_{0_+}^z \right) \left(w(z) - z^2 + \frac{1}{2}(z_1^2 + z_3^2) \right) dz = \\ & = 2i \oint_a \left(w(z) - z^2 + \frac{1}{2}(z_1^2 + z_3^2) \right) dz + \mathcal{O}(z^{-1}) = \frac{1}{2}A + \mathcal{O}(z^{-1}). \end{aligned}$$

Thus,

$$g(z) = i \left(\frac{4}{3}z^3 + e^{i\varphi}z \right) + \mathcal{O}(z^{-1}) \quad \text{as } z \rightarrow \infty, \quad (4.5)$$

and we have established the property a) of the function $g(z)$. The property c) of the function $g(z)$, i.e. $\text{Re } g(\pm z_{1,3}) = 0$, is equivalent to the system of equations $\text{Re } A = \text{Re } B = 0$ and thus takes the integral form,

$$\text{Im} \oint_{a,b} w(z) dz = 0. \quad (4.6)$$

Equations (4.6), known as the *Boutroux equations*, are the substantial ingredient of our proof below of the solvability of the RH problem.

Denote $D = z_1^2 z_3^2$ the module corresponding to our elliptic curve. We observe that (4.6) determines D as the differentiable function $D(\varphi)$ of the angle parameter $\varphi \in (\frac{2\pi}{3}, \pi)$. Indeed, the l.h.s. of (4.6) yields the complete set of the independent first integrals of the first order nonlinear ODE

$$\frac{dD}{d\varphi} = -\frac{1}{4\text{Im}(\omega_a \bar{\omega}_b)} (\bar{\omega}_a (e^{i\varphi} \Omega_a - e^{-i\varphi} \bar{\Omega}_a) - \omega_b (e^{i\varphi} \Omega_b - e^{-i\varphi} \bar{\Omega}_b)), \quad (4.7)$$

where

$$\omega_{a,b} = \oint_{a,b} \frac{dz}{w(z)}, \quad \Omega_{a,b} = \oint_{a,b} \frac{z^2 dz}{w(z)}.$$

Whenever the elliptic curve $w(z)$ does not degenerate, the r.h.s. of (4.7) is a smooth function of φ , and, given the initial point, equation (4.7) is uniquely solvable. To find the initial point, observe that equations (4.6) admit degeneration $z_1 = z_3$ for $\varphi = \pi$ and $z_1 = 0$ for $\varphi = \frac{2\pi}{3}$. At these boundary points, the r.h.s. of (4.7) remains continuous which ensures its solvability. The classical uniqueness theorem for ODEs breaks down at these boundary points, however the only one of the solutions of the differential equation (4.7) passing through the initial boundary point satisfies the integral equations (4.6).

Define the Stokes line for the linear equation (2.6) emanating from the point z_k , $k = 1, 2, 3, 4$, as the set of the points z satisfying the equation

$$\text{Im} \int_{z_k}^z w(z) dz = 0. \quad (4.8)$$

Define the Stokes graph as the union of all the Stokes lines. The conditions (4.6) mean that the Stokes graph is connected. On the other hand, conditions (4.6) allow us to choose the arcs of the Stokes lines as the segments $[z_3, z_1]$, $[z_1, -z_1]$ and $[-z_1, -z_3]$.

For the following, we need to know the jump conditions of the function $g(z)$. These are described by the equations

$$\begin{aligned} z \in (z_3, z_1): \quad g_+ + g_- &= \frac{1}{2}B; \\ z \in (z_1, -z_1): \quad g_+ - g_- &= -A; \\ z \in (-z_1, -z_3): \quad g_+ + g_- &= -\frac{1}{2}B. \end{aligned} \quad (4.9)$$

5 The model Baker-Akhiezer RH problem.

We remind that the reason why we replaced the original RH problem by the one presented in Figure 10 is that

$$e^{-tg(z)\sigma_3} S e^{tg(z)\sigma_3} \rightarrow I, \quad (5.1)$$

as $|t| \rightarrow \infty$ for all z belonging to the infinite branches of the jump contour $\hat{\gamma}$ and $z \neq \pm z_k$. Indeed, equation (5.1) means that when we normalize the RH problem, i.e. when we pass from the function Ψ to the function

$$Y(z) = \Psi(z)e^{tg(z)\sigma_3},$$

so that $Y(z) \rightarrow I$ as $z \rightarrow \infty$, the Y -jump matrices will exponentially fast approach the identity matrix as $|t| \rightarrow \infty$ everywhere on the infinite branches of $\hat{\gamma}$. Therefore, one can expect that the main contribution into the asymptotics of the solution of the RH problem will be made by the truncated RH problem posed on the sum of the segments (see Figure 6) $(z_3, z_1) \cup (z_1, -z_1) \cup (-z_1, -z_3)$. In other words, we arrive to the model problem consisting of finding the piecewise holomorphic function $\Psi^{(BA)}(z)$ with the following properties:

1) near infinity

$$\Psi^{(BA)}(z) = (I + \mathcal{O}(z^{-1}))e^{-t\theta\sigma_3}, \quad z \rightarrow \infty, \quad (5.2)$$

where $\theta = i(\frac{4}{3}z^3 + e^{i\varphi}z)$;

2) on the sum of the oriented contours $(z_3, z_1) \cup (z_1, -z_1) \cup (-z_1, -z_3)$, the jump conditions hold

$$z \in (-z_1, -z_3): \quad \Psi_+^{(BA)}(z) = \Psi_-^{(BA)}(z)S_A, \quad S_A = \begin{pmatrix} & -\frac{1}{s_3} \\ s_3 & \end{pmatrix}, \quad (5.3)$$

$$z \in (z_3, z_1): \quad \Psi_+^{(BA)}(z) = \Psi_-^{(BA)}(z)S_B, \quad S_B = \begin{pmatrix} & s_3 \\ -\frac{1}{s_3} & \end{pmatrix}, \quad (5.4)$$

$$z \in (z_1, -z_1): \quad \Psi_+^{(BA)}(z) = \Psi_-^{(BA)}(z)S_C, \quad S_C = \begin{pmatrix} \frac{1}{1-s_1s_3} & \\ & 1-s_1s_3 \end{pmatrix}; \quad (5.5)$$

3) the function $\Psi^{(BA)}(z)$ has no more than L_2 -integrable singularities at the points $\pm z_1, \pm z_3$.

We use the superscript (BA) since we are going to construct $\Psi^{(BA)}(z)$ in terms of the Baker-Akhiezer functions of the Riemann surface Γ . On this Riemann surface, there exists the only one, up to an arbitrary constant multiplier, holomorphic differential $\frac{dz}{w(z)}$ with the periods $\omega_{a,b}$. Introduce the canonical Abelian (elliptic) integral of the first kind $U(z)$,

$$U(z) = \frac{1}{\omega_a} \int_{\infty+}^z \frac{dz}{w(z)}, \quad (5.6)$$

and the Abelian integral $h(z)$ which differs from $g(z)$ (4.4) in some integral of the first kind,

$$h(z) = g(z) + cU(z), \quad (5.7)$$

where the constant c will be determined later. Since $U(z) = \mathcal{O}(z^{-1})$ as $z \rightarrow \infty$, the integral $h(z)$ has the same canonical asymptotics as the integral $g(z)$,

$$h(z) = i(\frac{4}{3}z^3 + e^{i\varphi}z) + \mathcal{O}(z^{-1}). \quad (5.8)$$

Denote the corresponding complete integrals by the letters A' and B' :

$$\begin{aligned} A' &= 4i \oint_a w(z) dz + \frac{c}{\omega_a} \oint_a \frac{dz}{w(z)} \equiv A + c, \\ B' &= 4i \oint_b w(z) dz + \frac{c}{\omega_a} \oint_b \frac{dz}{w(z)} \equiv B + c\tau, \quad \tau = \frac{\omega_b}{\omega_a}. \end{aligned} \quad (5.9)$$

Let us introduce the “intermediate” $\overset{o}{\Psi}$ -function,

$$\overset{o}{\Psi}(z) = \Psi^{(BA)}(z) e^{t_0 h(z) \sigma_3}, \quad (5.10)$$

where t_0 is the second reserved parameter to be defined later. The asymptotics of this function at infinity,

$$\overset{o}{\Psi}(z) = (I + \mathcal{O}(z^{-1})) e^{-(t-t_0)\theta \sigma_3}, \quad z \rightarrow \infty, \quad (5.11)$$

differs from the canonical in the difference $t - t_0$ which replaces the original t . Its jump properties are similar to those of the function $\Psi^{(BA)}$ except for they are described using the new parameters t_0 , A' and B' :

$$z \in (-z_1, -z_3): \quad \overset{o}{\Psi}_+(z) = \overset{o}{\Psi}_-(z) \hat{G}_A(z), \quad (5.12)$$

$$z \in (z_3, z_1): \quad \overset{o}{\Psi}_+(z) = \overset{o}{\Psi}_-(z) \hat{G}_B(z), \quad (5.13)$$

$$z \in (z_1, -z_1): \quad \overset{o}{\Psi}_+(z) = \overset{o}{\Psi}_-(z) \hat{G}_C(z), \quad (5.14)$$

where

$$\begin{aligned} \hat{G}_A &= \begin{pmatrix} & -s_3^{-1} e^{t_0 B'/2} \\ s_3 e^{-t_0 B'/2} & \end{pmatrix}, \quad \hat{G}_B = \begin{pmatrix} & s_3 e^{-t_0 B'/2} \\ -s_3^{-1} e^{t_0 B'/2} & \end{pmatrix}, \\ \hat{G}_C &= \begin{pmatrix} (1 - s_1 s_3)^{-1} e^{-t_0 A'} & \\ & (1 - s_1 s_3) e^{t_0 A'} \end{pmatrix}. \end{aligned}$$

Taking into account the conditions (3.11), we may choose the reserved parameters c and t_0 in such a way that

$$s_3 = e^{t_0 B'/2}, \quad 1 - s_1 s_3 = e^{-t_0 A'}. \quad (5.15)$$

The system (5.15) yields immediately the parameter values

$$\begin{aligned} t_0 &= -2 \frac{\omega_a}{\Delta} \ln s_3 - \frac{\omega_b}{\Delta} \ln(1 - s_1 s_3) \pmod{\frac{4\pi i \omega_a}{\Delta}} \pmod{\frac{2\pi i \omega_b}{\Delta}}, \\ \frac{c}{\omega_a} t_0 &= 2 \frac{A}{\Delta} \ln s_3 + \frac{B}{\Delta} \ln(1 - s_1 s_3) \pmod{\frac{4\pi i A}{\Delta}} \pmod{\frac{2\pi i B}{\Delta}}, \end{aligned} \quad (5.16)$$

where

$$\Delta = A\omega_b - B\omega_a = 4i(\mathcal{J}_a \omega_b - \mathcal{J}_b \omega_a) = -\frac{16\pi}{3}(z_1^2 + z_3^2) = \frac{8\pi}{3} e^{i\varphi}$$

due to the Riemann bilinear identity.

Under the conditions (5.15), the RH problem for the function $\overset{o}{\Psi}$ simplifies dramatically:

1) near infinity, it behaves like the exponent

$$\overset{o}{\Psi}(z) = (I + \mathcal{O}(z^{-1}))e^{-(t-t_0)\theta\sigma_3}, \quad z \rightarrow \infty; \quad (5.17)$$

2) it jumps on the intervals $(-z_1, -z_3)$ and (z_3, z_1) :

$$z \in (-z_1, -z_3): \quad \overset{o}{\Psi}_+(z) = \overset{o}{\Psi}_-(z) \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \quad (5.18)$$

$$z \in (z_3, z_1): \quad \overset{o}{\Psi}_+(z) = \overset{o}{\Psi}_-(z) \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}; \quad (5.19)$$

3) it has no more than L_2 -integrable singularities at $\pm z_1, \pm z_3$.

Apart from the countable set of values of t described below, this problem can be solved explicitly, cf. [7]. Let us introduce the function

$$\mu = \frac{1}{2} \begin{pmatrix} \beta + \beta^{-1} & -i(\beta - \beta^{-1}) \\ i(\beta - \beta^{-1}) & \beta + \beta^{-1} \end{pmatrix}, \quad (5.20)$$

where

$$\beta = \left(\frac{z^2 - z_1^2}{z^2 - z_3^2} \right)^{1/4} \quad (5.21)$$

is defined on the upper sheet of the Riemann surface Γ (see Figure 10) in such a way that

$$\beta \rightarrow 1 \quad \text{as} \quad z \rightarrow \infty,$$

so that

$$\mu \xrightarrow{z \rightarrow \infty} I. \quad (5.22)$$

The discontinuity of the scalar function β is described by the relations

$$\begin{aligned} z \in (-z_1, -z_3): \quad & \beta_+ = -i\beta_-, \\ z \in (z_3, z_1): \quad & \beta_+ = i\beta_-, \end{aligned}$$

and therefore, for the function μ , the jump conditions hold:

$$\begin{aligned} z \in (-z_1, -z_3): \quad & \mu_+ = \mu_- \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \\ z \in (z_3, z_1): \quad & \mu_+ = \mu_- \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \end{aligned} \quad (5.23)$$

The function $\overset{o}{\Psi}$ is constructed in terms of the Baker-Akhiezer functions as follows:

$$\overset{o}{\Psi}(z) = \begin{pmatrix} \mu_{11}^{(1)} \overset{(1)}{\Psi}_{BA} & \mu_{12}^{(1)} \overset{(1)}{\Psi}_{BA}^* \\ \mu_{21}^{(2)} \overset{(2)}{\Psi}_{BA} & \mu_{22}^{(2)} \overset{(2)}{\Psi}_{BA}^* \end{pmatrix} \quad (5.24)$$

where

$$\begin{aligned}
\Psi_{BA}^{(1)}(z) &= e^{-(t-t_0)\Omega(z)} \frac{\Theta(U(z) + V(t-t_0) + \frac{1}{2})\Theta(\frac{1}{2})}{\Theta(U(z) + \frac{1}{2})\Theta(V(t-t_0) + \frac{1}{2})}, \\
\Psi_{BA}^{(2)} &= e^{-(t-t_0)(\Omega(z) - \pi i V)} \frac{\Theta(U(z) + V(t-t_0) + \frac{1+\tau}{2})\Theta(\frac{1}{2})}{\Theta(U(z) + \frac{1+\tau}{2})\Theta(V(t-t_0) + \frac{1}{2})}, \\
\Psi_{BA}^{(1)*}(z) &= e^{(t-t_0)(\Omega(z) - \pi i V)} \frac{\Theta(U(z) - V(t-t_0) + \frac{1+\tau}{2})\Theta(\frac{1}{2})}{\Theta(U(z) + \frac{1+\tau}{2})\Theta(V(t-t_0) + \frac{1}{2})}, \\
\Psi_{BA}^{(2)*} &= e^{(t-t_0)\Omega(z)} \frac{\Theta(U(z) - V(t-t_0) + \frac{1}{2})\Theta(\frac{1}{2})}{\Theta(U(z) + \frac{1}{2})\Theta(V(t-t_0) + \frac{1}{2})}, \tag{5.25}
\end{aligned}$$

and

$$\Omega(z) = g(z) - AU(z), \quad U(z) = \frac{1}{\omega_a} \int_{\infty^+}^z \frac{dz}{w(z)}, \quad \tau = \frac{\omega_b}{\omega_a}, \tag{5.26}$$

$$\begin{aligned}
V &= \frac{\Delta}{2\pi i \omega_a} = \frac{8i}{3\omega_a} (z_1^2 + z_3^2) = -\frac{4i}{3\omega_a} e^{i\varphi}, \\
t &\neq t_0 + \frac{\tau}{2V} \pmod{\frac{1}{V}} \pmod{\frac{\tau}{V}}, \tag{5.27}
\end{aligned}$$

and $\Theta(z)$ means the Riemann theta-function of Γ , $\Theta(z) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n z}$.

It is worth to note that all $\Psi_{BA}^{(k)}(z)$, $\Psi_{BA}^{(k)*}(z)$ are single valued on \mathbb{C}^* cut along $[z_3, z_1] \cup [-z_1, -z_3]$, see Figure 10, because $\oint_a d\Omega = 0$. Also, considered on the Riemann surface, $\Psi_{BA}^{(k)}(z) = \Psi_{BA}^{(k)}(P)$ and $\Psi_{BA}^{(k)*}(z) = \Psi_{BA}^{(k)*}(P^*)$.

Now, let us verify that the function (5.24) yield the solution of the RH problem (5.17)–(5.18). Note first of all that $\overset{\circ}{\Psi}(z)$ has no pole on the upper sheet of Γ . Indeed, $\Theta(U(z) + \frac{1}{2}) \neq 0 \forall z \in \mathbb{C}^*$. Taking into account the equation $\Theta(\frac{1+\tau}{2}) = 0$, the only zero of $\Theta(U(z) + \frac{1+\tau}{2})$ lies at ∞^+ . Thus $\Theta(U(z) + \frac{1+\tau}{2}) = \mathcal{O}(z^{-1})$ while both μ_{12} and μ_{21} vanish as $\mathcal{O}(z^{-2})$.

Next, to check the asymptotics of $\overset{\circ}{\Psi}(z)$ near infinity on the upper sheet of Γ , we find

$$U(z) = -\frac{1}{\omega_a z} + \mathcal{O}(z^{-2}), \quad z \rightarrow \infty^+,$$

and therefore

$$\begin{aligned}
(\overset{\circ}{\Psi})_{11} &= e^{-i(t-t_0)(\frac{4}{3}z^3 + e^{i\varphi}z)} (1 + \mathcal{O}(z^{-1})), \\
(\overset{\circ}{\Psi})_{22} &= e^{i(t-t_0)(\frac{4}{3}z^3 + e^{i\varphi}z)} (1 + \mathcal{O}(z^{-1})).
\end{aligned}$$

Similarly, using the asymptotics

$$\mu_{12} = -i \frac{z_3^2 - z_1^2}{4z^2} + \mathcal{O}(z^{-4}), \quad \mu_{21} = i \frac{z_3^2 - z_1^2}{4z^2} + \mathcal{O}(z^{-4}),$$

$$\Theta\left(U(z) + \frac{1+\tau}{2}\right) = -\frac{1}{\omega_a z} \Theta'\left(\frac{1+\tau}{2}\right) + \mathcal{O}(z^{-2}),$$

we obtain

$$\left(\overset{o}{\Psi}\right)_{21} = e^{-i(t-t_0)(\frac{4}{3}z^3 + e^{i\varphi}z)} \cdot \mathcal{O}(z^{-1}), \quad \left(\overset{o}{\Psi}\right)_{12} = e^{i(t-t_0)(\frac{4}{3}z^3 + e^{i\varphi}z)} \cdot \mathcal{O}(z^{-1}),$$

and the asymptotics (5.17) holds true. To check the jump conditions (5.18), (5.19), it is enough, due to (5.23), to show that

$$\overset{(k)}{\Psi}_{BA}(z)_{\pm} = \overset{(k)}{\Psi}_{BA}^*(z)_{\mp}$$

on the segments (z_3, z_1) and $(-z_1, -z_3)$. Let us consider the first of them. One can see that on the segment (z_3, z_1) ,

$$U_+ + U_- = \frac{\tau}{2}, \quad \Omega_+ + \Omega_- = \frac{1}{2}(B - A\tau) = -\frac{\Delta}{2\omega_a} = -\pi i V.$$

Therefore

$$\begin{aligned} \overset{(1)}{\Psi}_{BA}(z)_+ &= e^{-(t-t_0)\Omega_+(z)} \frac{\Theta(U_+(z) + V(t-t_0) + \frac{1}{2})\Theta(\frac{1}{2})}{\Theta(U_+(z) + \frac{1}{2})\Theta(V(t-t_0) + \frac{1}{2})} = \\ &= e^{(t-t_0)(\Omega_-(z) + \pi i V)} \frac{\Theta(-U_-(z) + \frac{\tau}{2} + V(t-t_0) + \frac{1}{2})\Theta(\frac{1}{2})}{\Theta(-U_-(z) + \frac{\tau}{2} + \frac{1}{2})\Theta(V(t-t_0) + \frac{1}{2})} = \\ &= e^{(t-t_0)(\Omega_-(z) - \pi i V)} \frac{\Theta(U_-(z) + \frac{1+\tau}{2} - V(t-t_0))\Theta(\frac{1}{2})}{\Theta(U_-(z) + \frac{1+\tau}{2})\Theta(V(t-t_0) + \frac{1}{2})} = \overset{(1)}{\Psi}_{BA}^*(z)_-, \end{aligned}$$

where we have used the properties $\Theta(-z) = \Theta(z)$, $\Theta(z+1) = \Theta(z)$, $\Theta(z \pm \tau) = e^{-\pi i \tau \mp 2\pi i z} \Theta(z)$. The other conditions can be checked similarly.

Thus we obtain the following representation for the function $\Psi^{(BA)}$:

$$\begin{aligned} \Psi^{(BA)} &= \overset{o}{\Psi} e^{-t_0 h \sigma_3} = \begin{pmatrix} \overset{(1)}{\mu}_{11} \overset{(1)}{\Psi}_{BA} & \overset{(1)}{\mu}_{12} \overset{(1)}{\Psi}_{BA}^* \\ \overset{(2)}{\mu}_{21} \overset{(2)}{\Psi}_{BA} & \overset{(2)}{\mu}_{22} \overset{(2)}{\Psi}_{BA}^* \end{pmatrix} e^{-t_0 h \sigma_3} = \\ &= \begin{pmatrix} \overset{(1)}{\mu}_{11} \overset{(1)}{\Theta} & \overset{(1)}{\mu}_{12} e^{-\pi i V(t-t_0)} \overset{(1)}{\Theta}^* \\ \overset{(2)}{\mu}_{21} e^{\pi i V(t-t_0)} \overset{(2)}{\Theta} & \overset{(2)}{\mu}_{22} \overset{(2)}{\Theta}^* \end{pmatrix} e^{(tA - t_0 A') U \sigma_3} e^{-t g \sigma_3}, \end{aligned} \quad (5.28)$$

where

$$\begin{aligned} \overset{(1)}{\Theta} &= \frac{\Theta(U(z) + V(t-t_0) + \frac{1}{2})\Theta(\frac{1}{2})}{\Theta(U(z) + \frac{1}{2})\Theta(V(t-t_0) + \frac{1}{2})}, \quad \overset{(1)}{\Theta}^* = \frac{\Theta(U(z) - V(t-t_0) + \frac{1+\tau}{2})\Theta(\frac{1}{2})}{\Theta(U(z) + \frac{1+\tau}{2})\Theta(V(t-t_0) + \frac{1}{2})}, \\ \overset{(2)}{\Theta} &= \frac{\Theta(U(z) + V(t-t_0) + \frac{1+\tau}{2})\Theta(\frac{1}{2})}{\Theta(U(z) + \frac{1+\tau}{2})\Theta(V(t-t_0) + \frac{1}{2})}, \quad \overset{(2)}{\Theta}^* = \frac{\Theta(U(z) - V(t-t_0) + \frac{1}{2})\Theta(\frac{1}{2})}{\Theta(U(z) + \frac{1}{2})\Theta(V(t-t_0) + \frac{1}{2})}. \end{aligned}$$

Note finally, that the determinant of $\Psi^{(BA)}(z)$ might have singularities at the branch points $\pm z_1, \pm z_3$ of order $\mathcal{O}((z - z_k)^{-1/2})$. However, because of the asymptotics $\det \Psi^{(BA)}(z) = 1 + \mathcal{O}(z^{-1})$ as $z \rightarrow \infty$ and absence of jump, the determinant is entire function of z which, due to the Liouville theorem, is constant, $\det \Psi^{(BA)}(z) \equiv 1$.

6 Local RH problems near the branch points

6.1 RH problem solvable by the Airy functions

The function $\Psi^{(BA)}(z)$ has branch points at $z = z_k$, $k = 1, 2, 3, 4$, and does not approach the solution $\Psi(z)$ of the problem (3.28)–(3.35). In this section, we construct such an approximation using the model problem related to the Airy functions. Consider the Wronski matrices of the independent solutions $y_i(\zeta)$ of the Airy equation $y'' = \zeta y$:

$$\Phi(\zeta) = \begin{pmatrix} y_1(\zeta) & y_2(\zeta) \\ y_1'(\zeta) & y_2'(\zeta) \end{pmatrix}.$$

Using asymptotic formulae presented in ref. [2], one can introduce the matrix functions $\Phi_k(\zeta)$ with the standard asymptotic behavior as $\zeta \rightarrow \infty$,

$$\begin{aligned} \Phi_k(\zeta) &= \zeta^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} (I + \mathcal{O}(\zeta^{-3/2})) e^{\frac{2}{3}\zeta^{3/2}\sigma_3}, \quad \zeta^{3/2} \rightarrow \infty, \\ \zeta \in \omega_k &= \left\{ \zeta \in \mathbb{C} : \arg \zeta \in \left(\frac{2\pi}{3}(k - \frac{3}{2}) + \epsilon, \frac{2\pi}{3}(k + \frac{1}{2}) - \epsilon \right) \right\}, \quad \epsilon > 0. \end{aligned} \quad (6.1)$$

These functions are related to each other by the Stokes multipliers

$$\Phi_{k+1}(\zeta) = \Phi_k(\zeta) G_k, \quad G_{2k-1} = \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix}, \quad G_{2k} = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}, \quad (6.2)$$

This collection of the matrix functions allows us to construct an exact solution $\Phi(\zeta)$ of the following RH problem (see Figure 12):

1) near infinity

$$\Phi = \zeta^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} (I + \mathcal{O}(\zeta^{-3/2})) e^{\frac{2}{3}\zeta^{3/2}\sigma_3}, \quad \zeta^{3/2} \rightarrow \infty, \quad (6.3)$$

where $\zeta^{3/2}$ is defined on the plane cut along the ray $\arg \zeta = -\pi/3$;

2) on the anti-Stokes rays $\arg \zeta = \frac{2\pi}{3}k$, $k = 0, 1, 2$, oriented to infinity, the jump conditions hold,

$$\Phi_+ = \Phi_- G_k, \quad \arg \zeta = \frac{2\pi}{3}k, \quad k = 0, 1, 2; \quad (6.4)$$

3) on the ray $\arg \zeta = -\pi/3$ oriented to infinity, the jump condition holds,

$$\Phi_+ = \Phi_- G_\infty, \quad G_\infty = i\sigma_1 = \begin{pmatrix} & i \\ i & \end{pmatrix}. \quad (6.5)$$

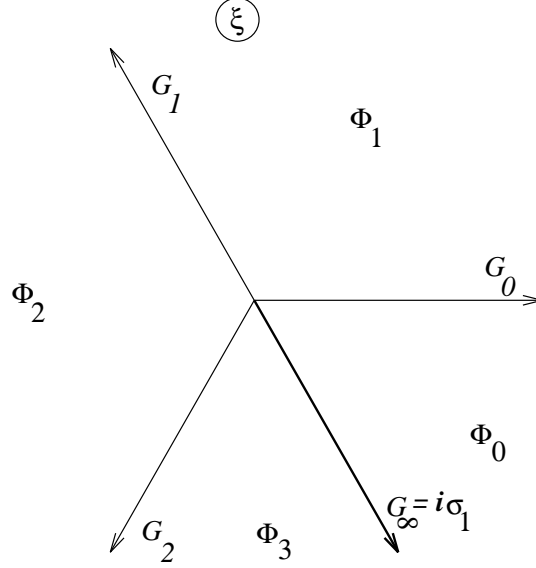


Figure 12: The RH problem graph for the collection of Wronski matrices of the Airy functions.

6.2 Parametrix in a neighborhood of the branch point z_3

Let us change the variable in accord with

$$\zeta(z) = \left(-\frac{3t}{2}g_3(z)\right)^{2/3}, \quad g_3(z) = 4i \int_{z_3}^z w(z) dz \equiv g(z) - \frac{1}{4}B. \quad (6.6)$$

This change is defined on the complex z -plane cut along the sum of intervals $[z_1, -z_1] \cup [-z_1, -z_3] \cup [-z_3, \infty)$ and mapping the domain \mathcal{D}_3 of z -plane bounded by the Stokes line chain shown in Figure 13 onto the complex ζ -plane cut along the ray $\arg \zeta = -\pi/3$. Now, the function

$$\check{\Phi}(z) = \Phi(\zeta(z)) \quad (6.7)$$

which behaves at infinity as follows,

$$\check{\Phi}(z) = \left(-\frac{3t}{2}g_3(z)\right)^{-\frac{\sigma_3}{6}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \left(I + \mathcal{O}\left(\frac{1}{tg_3(z)}\right)\right) e^{-tg_3(z)\sigma_3}, \quad tg_3(z) \rightarrow \infty, \quad (6.8)$$

jumps on the graph shown in Figure 13.

The rays $\arg \zeta = 2\pi k/3$, $k = 0, 1, 2$, are transformed into the anti-Stokes lines $\hat{\gamma}_5^{(3)}$, $\hat{\gamma}_6^{(3)}$ and $\hat{\gamma}_1^{(3)}$, respectively, see Figure 4, moreover the jump matrices are preserved there. The cut $\arg \zeta = -\pi/3$ turns out the segment (z_3, z_1) which splits at the point z_1 into the boundary of the z -domain.

It is important, that the jumps of the matrix functions $\Psi(z)$ and $\Phi^{(3)}(z)$ coincide with each other on the contours $\hat{\gamma}_5^{(3)}$, $\hat{\gamma}_6^{(3)}$, $\hat{\gamma}_1^{(3)}$ and (z_3, z_1) , so that they differ from each other in a left matrix multiplier $M^{(3)}(z)$ holomorphic in some bounded neighborhood of the node point $z = z_3$:

$$\Psi(z) = M^{(3)}(z)\Phi^{(3)}(z).$$

Besides the Airy fundamental matrix set $\Phi(\zeta)$, let us introduce its reduced asymptotic form

$$\Phi_{as}(\zeta) = \zeta^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} e^{\frac{2}{3}\zeta^{3/2}\sigma_3},$$

which solves the RH problem on the ray $\arg \zeta = -\pi/3$ oriented from zero to infinity,

$$\Phi_{as+} = \Phi_{as-} G_\infty, \quad G_\infty = i\sigma_1 = \begin{pmatrix} & i \\ i & \end{pmatrix}. \quad (6.12)$$

Using the changes (6.6), (6.9), the matrix

$$\Phi_{as}^{(3)}(z) = \Phi_{as}(\zeta(z)) e^{i\frac{\pi}{4}\sigma_3 s_3^{-\frac{\sigma_3}{2}}}$$

jumps on the Stokes segment (z_3, z_1) , and the jump matrix is (3.34)

$$\Phi_{as+}^{(3)} = \Phi_{as-}^{(3)} S_B, \quad S_B = \begin{pmatrix} & s_3 \\ -\frac{1}{s_3} & \end{pmatrix}, \quad z \in (z_3, z_1).$$

Since the jumps of the matrix functions $\Psi^{(BA)}(z)$ and $\Phi_{as}^{(3)}(z)$ coincide with each other on the segment (z_3, z_1) , the functions differ from each other in a left matrix multiplier $N^{(3)}(z)$ holomorphic in some bounded neighborhood of the point $z = z_3$,

$$\Psi^{(BA)}(z) = N^{(3)}(z)\Phi_{as}^{(3)}(z). \quad (6.13)$$

Let us define

$$\Psi^{(3)}(z) = N^{(3)}(z)\Phi^{(3)}(z), \quad N^{(3)}(z) = \Psi^{(BA)}(z)(\Phi_{as}^{(3)}(z))^{-1}. \quad (6.14)$$

6.3 Parametrix in a neighborhood of the branch point z_1

To construct the parametrix near $z = z_1$, let us introduce the function $\hat{\Phi}^{(1)}(\zeta)$,

$$\hat{\Phi}^{(1)}(\zeta) = \begin{cases} \Phi_0(\zeta) e^{-i\frac{\pi}{4}\sigma_3 \left(\frac{s_3}{1-s_1 s_3}\right)^{-\frac{\sigma_3}{2}}} & \text{for } \arg \zeta \in \left(-\frac{\pi}{3}, 0\right), \\ \Phi_1(\zeta) e^{-i\frac{\pi}{4}\sigma_3 \left(\frac{s_3}{1-s_1 s_3}\right)^{-\frac{\sigma_3}{2}}} & \text{for } \arg \zeta \in \left(0, \frac{2\pi}{3}\right), \\ \Phi_2(\zeta) e^{-i\frac{\pi}{4}\sigma_3 \left(\frac{s_3}{1-s_1 s_3}\right)^{-\frac{\sigma_3}{2}}} & \text{for } \arg \zeta \in \left(\frac{2\pi}{3}, \pi\right), \\ \Phi_2(\zeta) e^{-i\frac{\pi}{4}\sigma_3 (s_3(1-s_1 s_3))^{-\frac{\sigma_3}{2}}} & \text{for } \arg \zeta \in \left(\pi, \frac{4\pi}{3}\right), \\ \Phi_3(\zeta) e^{-i\frac{\pi}{4}\sigma_3 (s_3(1-s_1 s_3))^{-\frac{\sigma_3}{2}}} & \text{for } \arg \zeta \in \left(\frac{4\pi}{3}, \frac{5\pi}{3}\right). \end{cases} \quad (6.15)$$

As it is easy to see, the introduced matrix function satisfies the following jump conditions on the rays oriented to infinity,

$$\begin{aligned}
\hat{\Phi}_+^{(1)}(\zeta) &= \hat{\Phi}_-^{(1)}(\zeta) S_1^{(1)}, \quad S_1^{(1)} = \begin{pmatrix} 1 & 0 \\ -\frac{1-s_1 s_3}{s_3} & 1 \end{pmatrix}, \quad \arg \zeta = 0, \\
\hat{\Phi}_+^{(1)}(\zeta) &= \hat{\Phi}_-^{(1)}(\zeta) S_2^{(1)}, \quad S_2^{(1)} = \begin{pmatrix} 1 & \frac{s_3}{1-s_1 s_3} \\ 0 & 1 \end{pmatrix}, \quad \arg \zeta = \frac{2\pi}{3}, \\
\hat{\Phi}_+^{(1)}(\zeta) &= \hat{\Phi}_-^{(1)}(\zeta) S_C, \quad S_C = \begin{pmatrix} \frac{1}{1-s_1 s_3} & \\ & 1-s_1 s_3 \end{pmatrix}, \quad \arg \zeta = \pi, \\
\hat{\Phi}_+^{(1)}(\zeta) &= \hat{\Phi}_-^{(1)}(\zeta) S_5^{(1)}, \quad S_5^{(1)} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{s_3(1-s_1 s_3)} & 1 \end{pmatrix}, \quad \arg \zeta = \frac{4\pi}{3}, \\
\hat{\Phi}_+^{(1)}(\zeta) &= \hat{\Phi}_-^{(1)}(\zeta) S_B^{-1}, \quad S_B^{-1} = \begin{pmatrix} & -s_3 \\ \frac{1}{s_3} & \end{pmatrix}, \quad \arg \zeta = -\frac{\pi}{3}.
\end{aligned} \tag{6.16}$$

Introduce the change of variables

$$\zeta(z) = \left(-\frac{3t}{2}g_1(z)\right)^{2/3}, \tag{6.17}$$

where

$$g_1(z) = 4i \int_{z_1}^z w(z) dz$$

is defined on z -plane cut along $(\infty, z_3) \cup (z_3, z_1) \cup (\infty, -z_1) \cup (-z_1, -z_3)$. This change is holomorphic near the point z_1 , and hence it is defined on the z -plane cut along the sum of intervals $(z_3, \infty) \cup (\infty, -z_1) \cup (-z_1, -z_3)$ and maps the domain \mathcal{D}_1 of z -plane with the boundary on the Stokes lines shown on Figure 14, onto the complex ζ -plane cut along the rays $\arg \zeta = -\pi/3$, $\arg(\zeta - \zeta_2) = \pi$, with some ζ_2 , $\arg \zeta_2 = \pi$. Thus, in the area to the right of the cut $(z_3, z_1) \cup (z_1, -z_1)$, see Figure 13, or, equivalently, for $\arg(-\frac{3t}{2}g_1(z)) \in (-\frac{\pi}{2}, \frac{3\pi}{2})$,

$$g_1(z) = g(z) - \frac{1}{4}B - \frac{1}{2}A, \tag{6.18}$$

and in the area to the left of the cut $(z_3, z_1) \cup (z_1, -z_1)$ i.e. for z satisfying the condition $\arg(-\frac{3t}{2}g_1(z)) \in (\frac{3\pi}{2}, \frac{5\pi}{2})$,

$$g_1(z) = g(z) - \frac{1}{4}B + \frac{1}{2}A. \tag{6.19}$$

The function

$$\Phi^{(1)}(z) = \hat{\Phi}^{(1)}(\zeta(z)) \tag{6.20}$$

with the asymptotics at infinity

$$\begin{aligned}
\Phi^{(1)}(z) &= \left(-\frac{3t}{2}g_1(z)\right)^{-\frac{\sigma_3}{6}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \left(I + \mathcal{O}\left(\frac{1}{tg_1(z)}\right)\right) \times \\
&\times e^{-tg\sigma_3} e^{-i\frac{\pi}{4}\sigma_3} e^{\frac{1}{4}(tB-t_0B')\sigma_3} e^{\frac{1}{2}(tA-t_0A')\sigma_3},
\end{aligned}$$

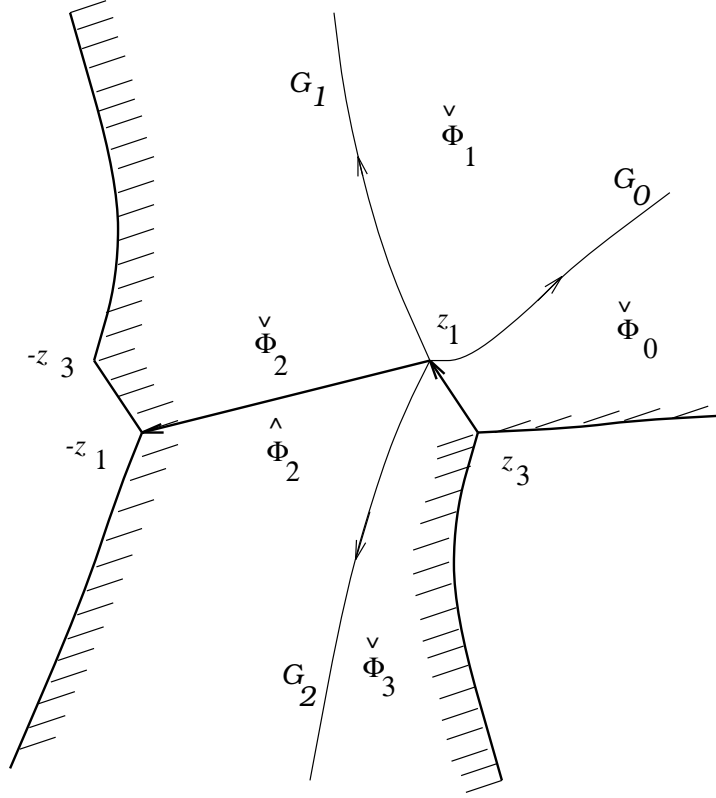


Figure 14: The jump graph for the set of the Wronski matrices of the Airy functions after the change of variables $\zeta = (-3tg_1(z)/2)^{2/3}$.

$$\begin{aligned}
tg_1(z) &\rightarrow \infty, \quad \arg(-tg_1(z)) \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right), \\
\Phi^{(1)}(z) &= \left(-\frac{3t}{2}g_1(z)\right)^{-\frac{\sigma_3}{6}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \left(I + \mathcal{O}\left(\frac{1}{tg_1(z)}\right)\right) \times \\
&\times e^{-tg\sigma_3} e^{-i\frac{\pi}{4}\sigma_3} e^{\frac{1}{4}(tB-t_0B')\sigma_3} e^{-\frac{1}{2}(tA-t_0A')\sigma_3}, \\
tg_1(z) &\rightarrow \infty, \quad \arg(-tg_1(z)) \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right),
\end{aligned} \tag{6.21}$$

jumps on the graph shown on Figure 14. Under the change (6.17), the interval (z_3, z_1) corresponds to a segment of the ray $\arg \zeta = -\frac{\pi}{3}$ (if the orientation of the ray is from infinity to the origin), the curve $\hat{\gamma}_1^{(1)}$ turns into the ray $\arg \zeta = 0$, the curve $\hat{\gamma}_2^{(1)}$ transforms into the ray $\arg \zeta = \frac{2\pi}{3}$, the Stokes line $(z_1, -z_1)$ corresponds to the ray $\arg \zeta = \pi$ and the curve $\hat{\gamma}_5^{(1)}$ corresponds to the ray $\arg \zeta = \frac{4\pi}{3}$.

Since the discontinuities of the functions $\Psi(z)$ and $\Phi^{(1)}(z)$ are the same on the contours $\hat{\gamma}_k^{(1)}$ and the Stokes segments (z_3, z_1) and $(z_1, -z_1)$, two these functions coincide with each other up to a left matrix multiplier $M^{(1)}(z)$ holomorphic in some bounded neighborhood of the point z_1 :

$$\Psi(z) = M^{(1)}(z)\Phi^{(1)}(z).$$

Besides the model function $\Phi^{(1)}(z)$, we introduce its reduced asymptotic form $\Phi_{as}^{(1)}(z)$ as follows:

$$\begin{aligned} \Phi_{as}^{(1)}(z) &= \left(-\frac{3t}{2}g_1(z)\right)^{-\frac{\sigma_3}{6}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \times \\ &\times e^{-tg\sigma_3} e^{-i\frac{\pi}{4}\sigma_3} e^{\frac{1}{4}(tB-t_0B')\sigma_3} e^{\frac{1}{2}(tA-t_0A')\sigma_3}, \\ tg_1(z) &\rightarrow \infty, \quad \arg(-tg_1(z)) \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right), \\ \Phi_{as}^{(1)}(z) &= \left(-\frac{3t}{2}g_1(z)\right)^{-\frac{\sigma_3}{6}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \times \\ &\times e^{-tg\sigma_3} e^{-i\frac{\pi}{4}\sigma_3} e^{\frac{1}{4}(tB-t_0B')\sigma_3} e^{-\frac{1}{2}(tA-t_0A')\sigma_3}, \\ tg_1(z) &\rightarrow \infty, \quad \arg(-g_1(z)) \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right), \end{aligned} \quad (6.22)$$

This reduced function is continuous across the anti-Stokes curves $\hat{\gamma}_k^{(1)}$, and is characterized by the jump conditions

$$\begin{aligned} \Phi_{as+}^{(1)}(z) &= \Phi_{as-}^{(1)}(z)S_C, \quad S_C = \begin{pmatrix} \frac{1}{1-s_1s_3} & \\ & 1-s_1s_3 \end{pmatrix}, \quad z \in (z_1, -z_1), \\ \Phi_{as+}^{(1)}(z) &= \Phi_{as-}^{(1)}(z)S_B, \quad S_B = \begin{pmatrix} & s_3 \\ -\frac{1}{s_3} & \end{pmatrix}, \quad z \in (z_3, z_1). \end{aligned}$$

Note again, that the jumps of the functions $\Psi^{(BA)}(z)$ and $\Phi_{as}^{(1)}(z)$ coincide on the segments (z_3, z_1) and $(z_1, -z_1)$ and hence the functions coincide with each other up to a left multiplier $N^{(1)}(z)$ holomorphic in some bounded neighborhood of the point z_1 :

$$\Psi^{(BA)}(z) = N^{(1)}(z)\Phi_{as}^{(1)}(z). \quad (6.23)$$

Define

$$\Psi^{(1)}(z) = N^{(1)}(z)\Phi^{(1)}(z), \quad N^{(1)}(z) = \Psi^{(BA)}(z)\left(\Phi_{as}^{(1)}(z)\right)^{-1}. \quad (6.24)$$

The functions solving the model RH problem associated with the branch points $-z_1$ and $-z_3$ can be constructed in the very same way, however they can be easily defined by the use of the symmetry relations:

$$\Psi^{(2)}(z) = \sigma_2\Psi^{(1)}(-z)\sigma_2, \quad \Psi^{(4)}(z) = \sigma_2\Psi^{(3)}(-z)\sigma_2. \quad (6.25)$$

7 Asymptotic solution of the main RH problem

In this section, we will construct the approximate solution $\tilde{\Psi}(z)$ of the RH problem (3.28)–(3.34) and prove the solvability of the latter. To this end, let us introduce four not overlapping disks B_k , $k = 1, 2, 3, 4$, centered at the branch points z_k , circles $C_k = \partial B_k$, and define the piecewise holomorphic function $\tilde{\Psi}(z)$,

$$\tilde{\Psi}(z) = \begin{cases} \Psi^{(k)}(z), & z \in B_k, \quad k = 1, 2, 3, 4, \\ \tilde{\Psi}(z) = \Psi^{(BA)}(z), & z \in \mathbb{C} \setminus \bigcup_{k=1}^4 B_k. \end{cases} \quad (7.1)$$

The function $\tilde{\Psi}$ solves the RH problem on the contour depicted on Figure 15:

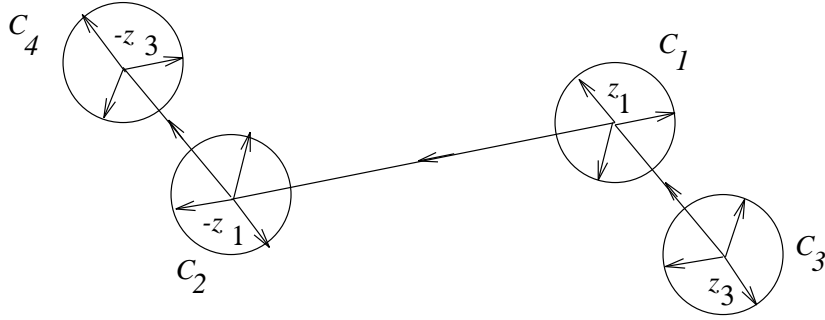


Figure 15: The model Riemann-Hilbert graph.

1) near infinity, due to (5.2),

$$\tilde{\Psi}(z) = \Psi^{(BA)}(z) = (I + \mathcal{O}(z^{-1}))e^{-t\theta\sigma_3}, \quad z \rightarrow \infty, \quad (7.2)$$

where $\theta = i(\frac{4}{3}z^3 + e^{i\varphi}z)$;

2) on the parts $\hat{\gamma}_s^{(k)} \cap B_k$ of anti-Stokes lines $\hat{\gamma}_s^{(k)}$ enclosed in the interior B_k of the circle C_k , and on the segments (z_3, z_1) , $(z_1, -z_1)$, $(-z_1, -z_3)$, the jump conditions hold

$$\tilde{\Psi}_+(z) = \tilde{\Psi}_-(z)S \quad (7.3)$$

with the jump matrices S defined in (3.30)–(3.34);

3) on the circles C_k , $k = 1, 2, 3, 4$, oriented counter-clockwise, the jump conditions hold

$$\tilde{\Psi}_+(z) = \tilde{\Psi}_-(z)H(z), \quad (7.4)$$

where, in accord with (6.13), (6.14), (6.23), (6.24), (2.5) and (6.25),

$$H(z) = (\Psi^{(BA)}(z))^{-1}\Psi^{(k)}(z) = (\Phi_{as}^{(k)}(z))^{-1}\Phi^{(k)}(z), \quad z \in C_k.$$

To describe the difference between the model problem solution $\tilde{\Psi}(z)$ and the solution $\Psi(z)$ of the original problem, let us construct the “ratio”

$$Z(z) = \Psi(z)\tilde{\Psi}^{-1}(z). \quad (7.5)$$

The correction function $Z(z)$ satisfies the RH problem on the jump contour shown on Figure 16:

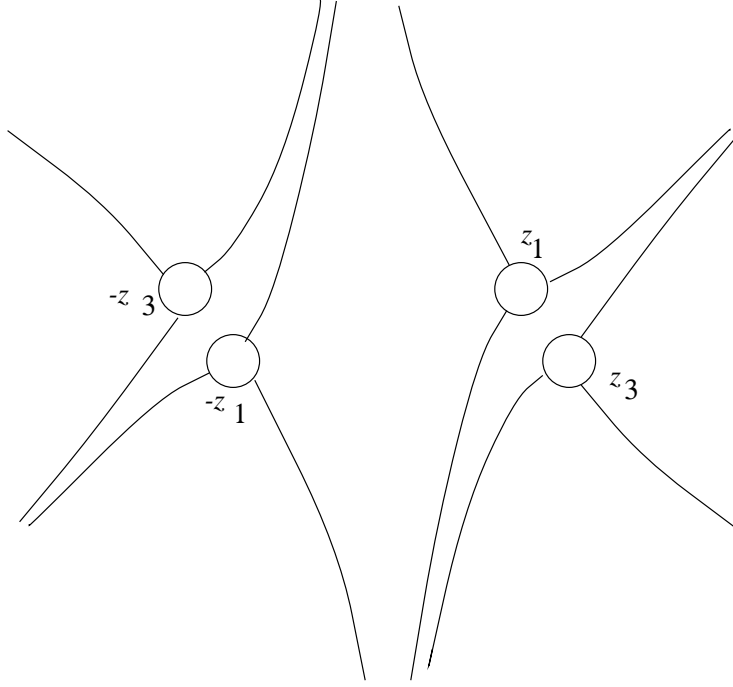


Figure 16: The RH graph for the correction function $Z(z)$.

1) $Z(z)$ is normalized at infinity

$$Z \xrightarrow{z \rightarrow \infty} I \quad (7.6)$$

2) on the parts of the anti-Stokes lines outside the circles C_k and on the circles themselves, the following jump conditions hold true:

$$Z_+(z) = Z_-(z)G_s^{(k)}(z), \quad z \in \hat{\gamma}_s^{(k)} \cap (\mathbb{C} \setminus B_k), \quad (7.7)$$

where, on the anti-Stokes tails,

$$G_s^{(k)}(z) = \Psi^{(BA)}(z)S_s^{(k)}(\Psi^{(BA)}(z))^{-1},$$

and, on the circles C_k , $k = 1, 2, 3, 4$,

$$Z_+(z) = Z_-(z)G_k(z), \quad z \in C_k, \quad (7.8)$$

where

$$G_k(z) = \Psi^{(BA)}(z)(\Psi^{(k)}(z))^{-1} =$$

$$= N^{(k)}(z) \Phi_{as}^{(k)}(z) (\Phi^{(k)}(z))^{-1} (N^{(k)}(z))^{-1}.$$

As it is easy to see, for t lying apart from the points (5.27), the jump matrices on the anti-Stokes lines have the asymptotics

$$G_s^{(k)}(z) = I + (e^{-2|\operatorname{Re}(tg(z))|}) = I + (e^{-2|\operatorname{Re}(tg_k(z))|}),$$

where we have used the Boutroux equations (4.6). For t lying apart from the points (5.27), using the definition of $\Phi_{as}^{(k)}(z)$ and the boundedness of the matrices $N(k)(z)$ and their inverse, we find the estimate

$$G_k(z) = I + (t^{-1}g_k^{-1}(z)).$$

Now, we are going to prove the solvability of the RH problem (7.6)–(7.8).

First note that the jump matrices are continuous on the RH problem graph. Consider the connected component of the graph associated with the point z_3 (see Figure 17).

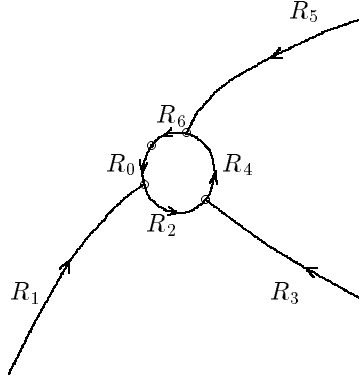


Figure 17: The connected component of the RH problem graph associated to the point z_3 .

Seeking for convenience, we orient the anti-Stokes lines $\hat{\gamma}_s^{(3)}$ in the direction from infinity to z_3 . Denote provisionally the jump matrices for the function $Z(z)$ as $R(z)$, i.e. $Z_+(z) = Z_-(z)R(z)$, moreover:

on the part $C_{25}^{(3)}$ of the circle C_3 between the points of its intersection with the Stokes line (z_3, z_1) and anti-Stokes line $\hat{\gamma}_5^{(3)}$, the jump matrix $R(z) = R_0(z)$ is as follows:

$$R_0(z) = N^{(3)}(z) \Phi_{as}^{(3)}(z) (\Phi_0^{(3)}(z))^{-1} (N^{(3)}(z))^{-1} = I + \mathcal{O}(t^{-1}g_3^{-1}(z)); \quad (7.9)$$

on the anti-Stokes line $\hat{\gamma}_5^{(3)}$, the jump matrix $R(z) = R_1(z)$ is given by

$$R_1(z) = \Psi^{(BA)}(z) (S_5^{(3)})^{-1} (\Psi^{(BA)}(z))^{-1} = I + \mathcal{O}(e^{2tg(z)}); \quad (7.10)$$

on the part $C_{56}^{(3)}$ of the circle C_3 between the points of its intersection with the anti-Stokes lines $\hat{\gamma}_5^{(3)}$ and $\hat{\gamma}_6^{(3)}$, the jump matrix $R(z) = R_2(z)$ is as follows:

$$\begin{aligned} R_2(z) &= N^{(3)}(z)\Phi_{as}^{(3)}(z)(\Phi_1^{(3)}(z))^{-1}(N^{(3)}(z))^{-1} = \\ &= N^{(3)}(z)\Phi_{as}^{(3)}(z)(S_5^{(3)})^{-1}(\Phi_0^{(3)}(z))^{-1}(N^{(3)}(z))^{-1} = \\ &= R_1(z)R_0(z) = I + \mathcal{O}(t^{-1}g_3^{-1}(z)); \end{aligned} \quad (7.11)$$

on the anti-Stokes line $\hat{\gamma}_6^{(3)}$, the jump matrix $R(z) = R_3(z)$ is given by

$$R_3(z) = \Psi^{(BA)}(z)(S_6)^{-1}(\Psi^{(BA)}(z))^{-1} = I + \mathcal{O}(e^{-2tg(z)}); \quad (7.12)$$

on the part $C_{61}^{(3)}$ of the circle C_3 between the points of its intersection with the anti-Stokes lines $\hat{\gamma}_6^{(3)}$ and $\hat{\gamma}_1^{(3)}$, the jump matrix $R(z) = R_4(z)$ is as follows:

$$\begin{aligned} R_4(z) &= N^{(3)}(z)\Phi_{as}^{(3)}(z)(\Phi_2^{(3)}(z))^{-1}(N^{(3)}(z))^{-1} = \\ &= N^{(3)}(z)\Phi_{as}^{(3)}(z)(S_6)^{-1}(S_5^{(3)})^{-1}(\Phi_0^{(3)}(z))^{-1}(N^{(3)}(z))^{-1} = \\ &= R_3(z)R_1(z)R_0(z) = I + \mathcal{O}(t^{-1}g_3^{-1}(z)); \end{aligned} \quad (7.13)$$

on the anti-Stokes line $\hat{\gamma}_1^{(3)}$, the jump matrix $R(z) = R_5(z)$ is given by

$$R_5(z) = \Psi^{(BA)}(z)(S_1^{(3)})^{-1}(\Psi^{(BA)}(z))^{-1} = I + \mathcal{O}(e^{-2tg(z)}); \quad (7.14)$$

on the part $C_{12}^{(3)}$ of the circle C_3 between the points of its intersection with the anti-Stokes line $\hat{\gamma}_1^{(3)}$ and the Stokes line (z_3, z_1) , the jump matrix $R(z) = R_6(z)$ is as follows:

$$\begin{aligned} R_6(z) &= N^{(3)}(z)\Phi_{as}^{(3)}(z)(\Phi_3^{(3)}(z))^{-1}(N^{(3)}(z))^{-1} = \\ &= N^{(3)}(z)\Phi_{as}^{(3)}(z)(S_1^{(3)})^{-1}(S_6)^{-1}(S_5^{(3)})^{-1}(\Phi_0^{(3)}(z))^{-1}(N^{(3)}(z))^{-1} = \\ &= R_5(z)R_3(z)R_1(z)R_0(z) = I + \mathcal{O}(t^{-1}g_3^{-1}(z)). \end{aligned} \quad (7.15)$$

The relations

$$R_2 = R_1R_0, \quad R_4 = R_3R_2, \quad R_6 = R_5R_4 \quad (7.16)$$

above provide us with the continuity of the RH problem at the node points $C_3 \cap \hat{\gamma}_s^{(3)}$ (see [27]). The fact is obvious because of the possibility of analytical continuation of the jump matrices $R(z)$ from the contours C_3 , $\hat{\gamma}_s^{(3)}$ and the possibility to split the graph at the node points in accord with (7.16) and to transform it into the union of the circle C_3 and of three smooth curves with the common endpoint $P = C_3 \cap (z_3, z_1)$. As to the point P itself, the continuity of the RH problem follows from the equation $R_0(z) = R_6(z)$. Indeed, at this point,

$$\begin{aligned} R_0(z) &= \Psi_+^{(BA)}(z)(\Phi_0^{(3)}(z))_+^{-1}(N^{(3)}(z))^{-1} = \\ &= \Psi_-^{(BA)}(z)S_B(\Phi_3^{(3)}(z)S_B)_+^{-1}(N^{(3)}(z))^{-1} = \\ &= \Psi_-^{(BA)}(z)(\Phi_3^{(3)}(z))_+^{-1}(N^{(3)}(z))^{-1} = R_6(z), \end{aligned} \quad (7.17)$$

where plus and minus mean the left and right limits on the segment (z_3, z_1) . In the very similar manner, one can check the continuity of the RH problem on the other connected components of the graph.

All the jump matrices are uniformly close to identity. More exactly, for t lying apart from the points (5.27), the estimates hold true,

$$\|R(z) - I\| < \begin{cases} c_1 e^{-2|\operatorname{Re}(tg(z))|}, & z \in \hat{\gamma}_s^{(k)}, \\ c_1 |t|^{-1} |g_k(z)|^{-1}, & z \in C_k, \end{cases} \quad (7.18)$$

where the concrete value of the constant c_1 is not important for us. The estimates (7.18) imply that the differences $R(z) - I$ are square integrable on the RH problem graph $\Sigma = \cup_k C_k \cup_{k,s} \hat{\gamma}_s^{(k)}$, i.e. $R(z) - I \in L^2(\Sigma)$.

The RH problem (7.6)–(7.8), i.e.

$$\begin{aligned} Z &\xrightarrow{z \rightarrow \infty} I, \\ Z_+(z) &= Z_-(z)R(z), \quad z \in \Sigma, \end{aligned} \quad (7.19)$$

is equivalent to the system of singular integral equations for the function $\rho(\zeta) \equiv Z_-(\zeta)$,

$$\rho(z) = I + \frac{1}{2\pi i} \int_{\Sigma} \rho(\zeta)(R(\zeta) - I) \frac{d\zeta}{\zeta - z_-}, \quad (7.20)$$

where z_- denotes the right limit of z on the contour of integration. In the symbolic operator form, the system reads

$$\rho = I + C_- [M\rho], \quad (7.21)$$

where C_- is the Cauchy operator, and M is the operator of the right multiplication in the matrix $R(\zeta) - I \in L^2(\Sigma)$. Thus M acts in $L^2(\Sigma)$ and satisfies the estimate

$$\|M\|_{L^2(\Sigma)} < c_2 |t|^{-1}, \quad (7.22)$$

where the precise value of the constant c_2 is not important for us. Because the Cauchy operator is bounded in $L^2(\Sigma)$, see [24, 3, 27], we obtain the estimate

$$\|C_- M\|_{L^2(\Sigma)} < c_3 |t|^{-1}. \quad (7.23)$$

Therefore, for any large enough t lying apart from the points (5.27), the operator $C_- M$ is contracting, equation for the function $\chi = \rho - I$,

$$\chi = C_- [M] + C_- [M\chi], \quad (7.24)$$

is solvable in the space $L^2(\Sigma)$ by iterations, and the estimate holds true,

$$\|\rho - I\|_{L^2(\Sigma)} < c |t|^{-1}. \quad (7.25)$$

The solution of the RH problem (7.19) for the function $Z(z)$ is given by the Cauchy integral

$$Z(z) = I + \frac{1}{2\pi i} \int_{\Sigma} \rho(\zeta)(R(\zeta) - I) \frac{d\zeta}{\zeta - z},$$

which implies the asymptotic relation

$$\Psi(z) = (I + \mathcal{O}((1 + |z|)^{-1}t^{-1}))\tilde{\Psi}(z) \quad (7.26)$$

in any closed subdomain of $\mathbb{C} \setminus \Sigma$.

8 Asymptotics of the Painlevé function

Now we are ready to compute the asymptotics of the Painlevé function. Using the expansion of $\Psi(\lambda)$ near infinity,

$$\begin{aligned} \Psi(\lambda) &= (I + \frac{1}{\lambda}(-\frac{i}{2}D\sigma_3 + \frac{1}{2}u\sigma_1) + \mathcal{O}(\frac{1}{\lambda^2}))e^{-i(\frac{4}{3}\lambda^3 + x\lambda)\sigma_3} = \\ &= (I + \frac{1}{t^{1/3}z}(-\frac{i}{2}D\sigma_3 + \frac{1}{2}u\sigma_1) + \mathcal{O}(\frac{1}{t^{2/3}z^2}))e^{-it(\frac{4}{3}z^3 + e^{i\varphi}z)\sigma_3}, \end{aligned} \quad (8.1)$$

where u is the Painlevé function (cf. (2.7)), i.e. the solution of (1.1), $D = v^2 - u^4 - xu^2$ and

$$\Psi(z) = (I + \mathcal{O}(\frac{1}{tz}))\tilde{\Psi}(z) = (I + \mathcal{O}(t^{-1}z^{-1}))\overset{\circ}{\Psi}(z)e^{-t_0h(z)\sigma_3}, \quad (8.2)$$

to find the asymptotics of u up to terms of order t^{-1} , it is enough to expand the Baker-Akhiezer function $\overset{(1)}{\Psi}_{BA}^*$ or $\overset{(2)}{\Psi}_{BA}$ near infinity:

$$\begin{aligned} \frac{u}{2t^{1/3}} &= \lim_{z \rightarrow \infty} ze^{\pi i(t-t_0)V} \frac{\Theta(U(z) + V(t-t_0) + \frac{1+\tau}{2})\Theta(\frac{1}{2})}{\Theta(U(z) + \frac{1+\tau}{2})\Theta(V(t-t_0) + \frac{1}{2})}\mu_{21} + \mathcal{O}(t^{-1}) = \\ &= i\frac{\omega_a}{4}(z_1^2 - z_3^2)e^{\pi i(t-t_0)V} \frac{\Theta(V(t-t_0) + \frac{1+\tau}{2})\Theta(\frac{1}{2})}{\Theta(V(t-t_0) + \frac{1}{2})\Theta'(\frac{1+\tau}{2})} + \mathcal{O}(t^{-1}). \end{aligned} \quad (8.3)$$

Equations (4.6), (5.16), and (8.3) are transformed to the statement of Theorem 2.1 via routine manipulations with the elliptic functions and integrals.

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